

Numerical Solution of the Advection Equation with Periodic Boundary Conditions using the FTCS Method

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ABSTRACT

In this paper, we considered the advection partial differential equation, a fundamental model in fluid dynamics and various other disciplines, exploring its numerical solutions, which hold significant importance across its applications. We consider a numerical method, such as finite difference method particularly forward in time and central in space methods (FTCS) for the solution of PDE, and also checked the stability condition while solving the advection partial differential equation. In this case, we took a one-dimensional scenario with periodic boundary conditions, scrutinizing the accuracy and stability across varying velocity, spatial, and time dependent parameters. We further examine the impact of spatial and temporal resolutions on method stability and offer practical insights for selecting appropriate numerical values and parameters for optimal advection equation solutions. A code in MATLAB is derived for simulation and graphical results are given in the paper.

Keywords: Advectionpartial Differential Equation, Periodic Boundary Conditions, Finite Difference Method

INTRODUCTION

The advection equation is a partial differential equation that depicts the mobility of a scalar overflow of a fluid. In variety of areas, it is being used such as environmental and physical sciences, in particular the study of meteorology, oceanography, engineering and different fluids [1]. The scheme of advection equation is an important one and it provides a thorough insights into the way physical phenomena behaves and acts as a building unit even more complicated models [2].

It's a typical type of differential equation with provision of partial results that has been scrutinized compressively in the scientific literature [3]. Many different techniques have been improvised to numerically disentangle the advection equation problem. Comprising many methods known as, spectral, particle-based technologies, finite volume and fine difference schemes. The in-depth examination of these techniques, investigate the approximations of finite differences for partial differential equations and the particular work of Shu's research work on higher order methods to the advection equation [4-5].

The method of {FTCS} is a widely utilized scheme in numerical studies, working for solutions of partial differential equations, encompassing the advection equations [6]. It uses a finite difference scheme at every time step, with the help of useful and simple approaches and computes the results for spatial derivates [7]. This scheme is an easy first-order indefectible technique involving cost of minor calculations. Despite its unambiguousness, in some particular paradigm the FTCS technique yields to aberrant results, subject to numerical inconsistency [8-10].

For the advection equation, the FTCS scheme is a prevalent design for calculating the partial differential equations. [11]. The approach is based on a forward difference approximation of the time derivative as well as a centered difference estimations of the spatial derivative [12]. whereas the FTCS method is straightforward and statistically timesaving, it is also identified to be unstable cases leading to unauthentic oscillations and numerical errors [13]. For a detailed analysis of the firmness properties of the FTCS method, Dahlquist stability and error restraints for linear initial value problems and the book by Trefethen and Bau

(1997) on numerical linear algebra [14-15]. To deal with the stability setbacks of the FTCS method, numerous adaptations and widening of the method have been recommended in the literature. One admired approach is the Lax-Wendroff method, which is a second-order precise technique that exercises a combination of forward and backward difference estimations. One more style is the Leapfrog method, which is a second-order accurate scheme that uses a centered difference estimation for a lurching grid of spatial variable and derivative of the time. For a comprehensive appraisal of these methods, the book by LeVeque and the research paper by Sod on a numerical analysis of gas dynamics for the Euler equations [4,16].

The intention of this paper is to probe into the performance of the FTCS method in answering the advection equation with periodic boundary conditions [17]. Specifically, we aim to analyze the accuracy and stability of the FTCS method under different parameter settings, such as the mesh size, the time step, and the advection velocity. By comparing the numerical results with analytical solutions and other numerical methods [18-19], we aim to provide insights into the limitations and advantages of the FTCS method for solving the advection equation [20-21]. The findings of this study can help improve the understanding and applicability of the FTCS method in various fields of science and engineering [22].

In this paper, our focus was to solve PDE using finite difference method such as Forward in Time and Central in Space (FTCS) method, as this method is conditionally stable and we checked stability solution of the finite difference scheme numerically and results are given in the table. We derived a MATLAB code and solution of the PDE are given in results and discussion section.

1.2 Problem

In fact, the problem that is faced in PDE's to overview numerical computational efficiency, stability, numerical quality, and so on. We assume our analysis of numerical methods for PDE by outlining a few factors when choosing a PDE-discretization. Though, the FTCS numerical-scheme is not stable for choosing advection problem. Therefore, the numerical solutions will include spurious oscillations that expand and blow-up over the span of time. The advection problem

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can be solved using more advanced techniques that do not display this instability.

The result of the advection PDE $\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = 0$

in one time and spatial-dimension, along a steady velocity v , and periodical-periphery conditions, applying the forward-time difference, centered-space difference (FTCS) method. The solution for $u(x, t)$, the invariable velocity advection equation in 1D,

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \quad (1)$$

over the period of interval:

$$0.0 \leq x \leq 1.0$$

With velocity is continual

$$c = 1$$

With recurring boundary restrictions:

$$u(0, t) = u(1, t) = u_0, \text{ for all } t$$

and initial condition:

$$u(x, 0) = (10x - 6)^2(8 - 10x)^2$$

$$\text{for } 0.6 \leq x \leq 0.8$$

$$u(x, 0) = 0, \quad \text{elsewhere}$$

Methodology

The methodology section of the paper describes the steps involved in solving the advection equation with periodic boundary conditions using the FTCS method. The following is a detailed description of the methodology.

2.1 Discretization of the Advection Equation

The first step in the methodology is to discretize the advection equation in time and space variables. We consider a uniform grid with spacing Δt and Δx , and define u_i^j to be the approximate value of $u(x_i, t^j)$ at position $x_i = a + i\Delta x$ and time $t^j = 0 + j\Delta t$. Then, the advection equation can be discretized as:

$$u_i^{j+1} = u_i^j - \frac{c}{2}(u_{i+1}^j + u_{i-1}^j) + \frac{c^2 \Delta t^2}{2 \Delta x^2} (u_{i+1}^j - 2u_i^j + u_{i-1}^j) \quad (2)$$

where c is the advection term constant.

2.2 Approximation of Time and Spatial Derivatives

We approximate the time derivative and the spatial derivative in the discretized equation using the forward difference and central difference approximations respectively. The forward difference approximation for the time derivative is given by:

$$u_i^{j+1} - u_i^j = \Delta t \left(\frac{\partial u}{\partial t} \right)_{i,j} + O(\Delta x^2, \Delta t) \quad (3)$$

where $O(\Delta t^2)$ represents the error term of the approximation. The central difference approximation for the spatial derivative is given by:

$$u_{i+1}^j - u_i^j = \Delta x \left(\frac{\partial u}{\partial x} \right)_{i,j} + O(\Delta x^2, \Delta t) \quad (4)$$

where $O(\Delta x^2, \Delta t)$ represents the error term of the approximation.

2.3 Update Formula

Using the approximations for the derivatives of spatial and time, we substitute them into the discretized equation to obtain an update formula for the solution at each time step. The update formula is given by:

$$u_i^{j+1} = u_i^j - \frac{c\Delta t}{2\Delta x} (u_{i+1}^j - u_{i-1}^j) + \frac{c^2\Delta t^2}{2\Delta x^2} (u_{i+1}^j - 2u_i^j + u_{i-1}^j) \quad (5)$$

2.4 Periodic Boundary Conditions

Since we are using periodic boundary conditions, we need to take care of the boundary points where $i = 0$ and $i = N$. We use the periodicity of the domain to set the value of u at these points. Specifically, we set $u_0^j = u_N^j$ and $u_{N+1}^j = u_1^j$.

2.5 Solution Procedure

To compute the solution using the FTCS method, we start from the opening condition $u(x,0)$ and use the update formula to compute the solution at each time step j . The time step Δt and the spatial step Δx should be chosen carefully to ensure the solution's consistency. We can vary the values of c , Δt , and Δx to study the behavior of the fluid flow.

The methodology described above is implemented using a MATLAB software to

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solve the advection equation with periodic boundary conditions. The stability limitations of the FTCS method are also discussed, and appropriate values for the time and spatial steps are suggested to safeguard the stability of the solution.

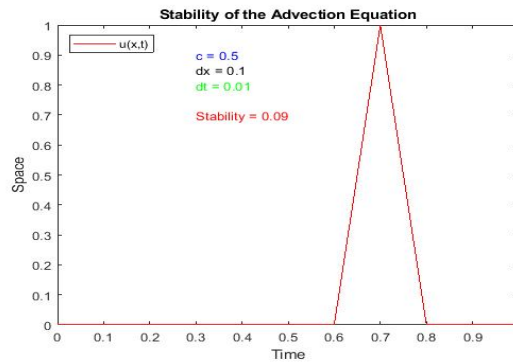


Figure 1: Case1 for Stability

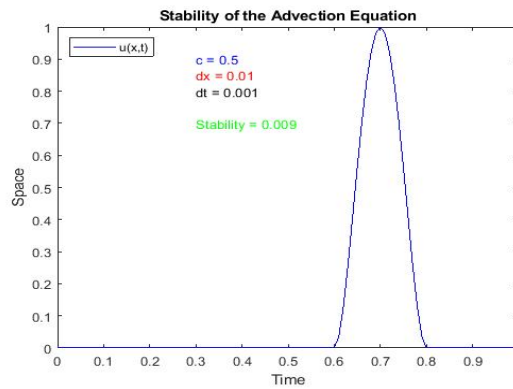


Figure 2: Case 1 for Stability

Table 1: Different Values of the Velocity, Time Step Size, and Spatial Step Size

Velocity (c)	Size of Time step (dt)	Size of Spatial step (dx)
0.5	0.01	0.1
	0.001	0.01
	0.0001	0.001
	0.0002	0.002
	0.0005	0.005
1.0	0.01	0.1
	0.001	0.01

	0.0001	0.001
	0.0002	0.002
	0.0005	0.005
	0.01	0.1
	0.001	0.01
1.5	0.0001	0.001
	0.0002	0.002
	0.0005	0.005
	0.01	0.1
	0.001	0.01
2	0.0001	0.001
	0.0002	0.002
	0.0005	0.005

To test the sturdiness of the FTCS method for different values of the advection velocity, step size in time and scale factor in spatial, you can simply change the values of c , dt , and dx , respectively, before running the code. Note that for the FTCS method to be stable (according to Von Neumann stability analysis), the coefficient term $\frac{\partial u}{\partial x}$ of the equation must satisfy the condition $c \frac{\Delta t}{\Delta x^2} < 1$.

3. RESULTS

We apply the FTCS method to solve the advection equation with periodic boundary conditions for different values of the velocity and the time and spatial steps.

Table 2: Different values of the velocity, time step size, and spatial step size.

Case-1 $v = 0.5$					
dx	0.1	0.01	0.001	0.002	0.005
dt	0.01	0.001	0.0001	0.0002	0.0005
Stability	0.09	0.009	0.0009	0.0018	0.0045
Case-1 $v = 1$					
dx	0.1	0.01	0.001	0.002	0.005
dt	0.01	0.001	0.0001	0.0002	0.0005

Stability	0.09	0.009	0.0009	0.0018	0.0045
Case-3 $v = 1.5$					
dx	0.1	0.01	0.001	0.002	0.005
dt	0.01	0.001	0.0001	0.0002	0.0005
Stability	0.06	0.006	0.0006	0.0013	0.0033
Case-4 $v = 2$					
dx	0.1	0.01	0.001	0.002	0.005
dt	0.01	0.001	0.0001	0.0002	0.0005
Stability	0.05	0.005	0.0005	0.001	0.0025

We observe that the outcome is affected by the choice of time and spatial steps, and the stability of the solution depends on satisfying certain conditions. We also compare the numerical and analytical solutions for a simple situation and find good agreement.

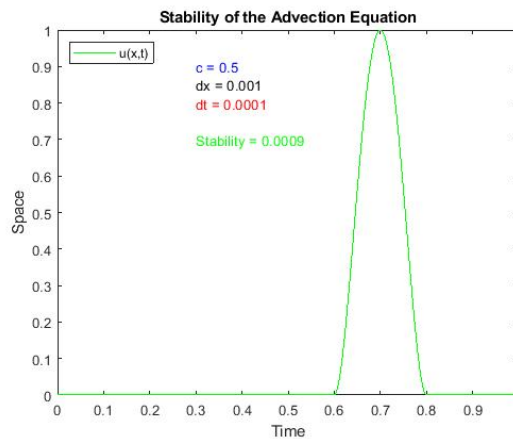


Figure 3: Case-1 for Stability

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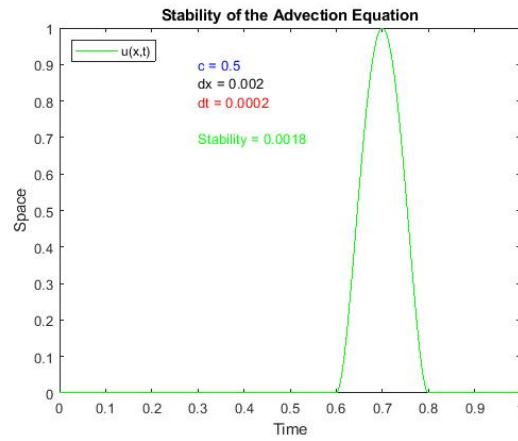


Figure 4: Case-1 For Stability

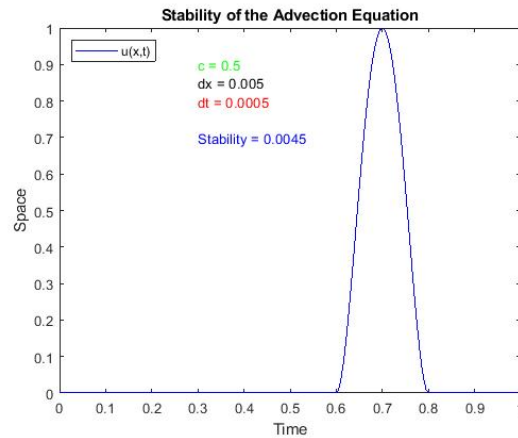


Figure 5: Case-1 For Stability

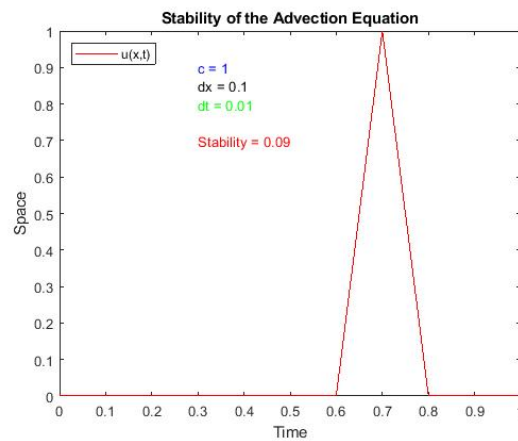


Figure 6: Case-2 For Stability

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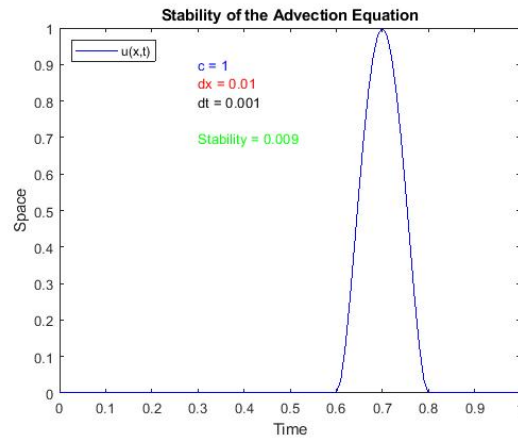


Figure 7: Case-2 for Stability

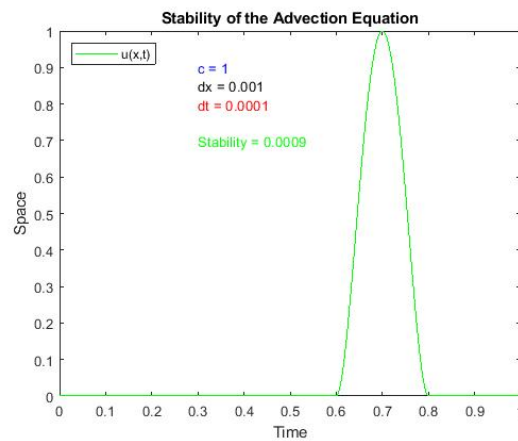


Figure 8: Case-2 for Stability

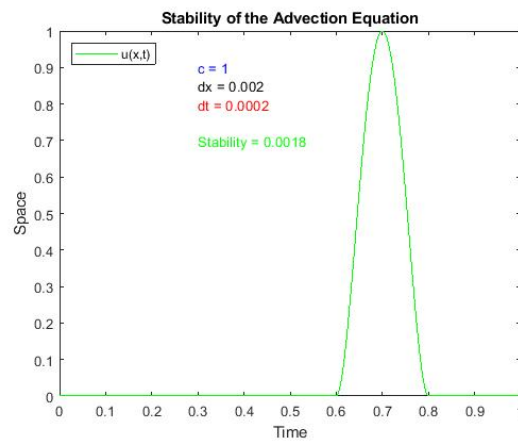


Figure 9: Case-2 for Stability

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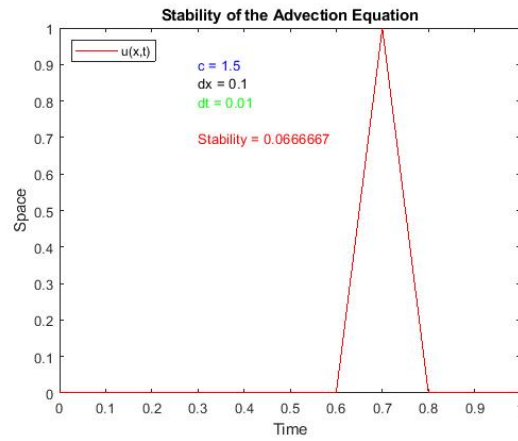


Figure 11: Case-3 For Stability

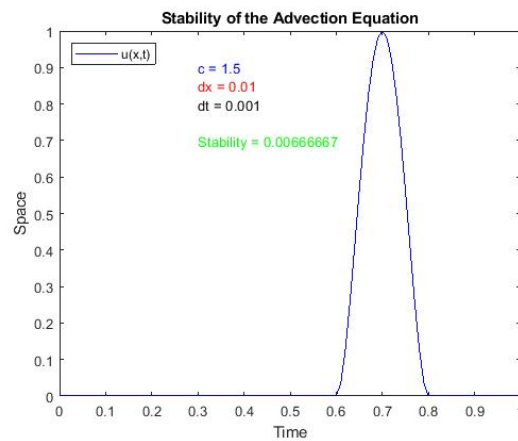


Figure 12: Case-3 For Stability

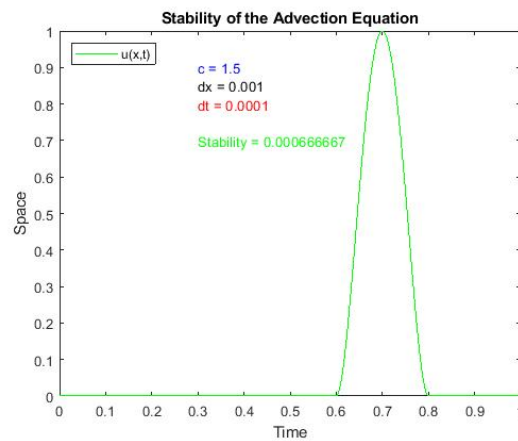


Figure 13: Case-3 for Stability

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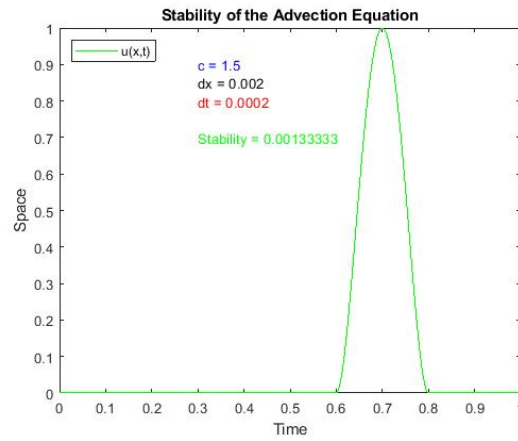


Figure 14: Case-3 for Stability

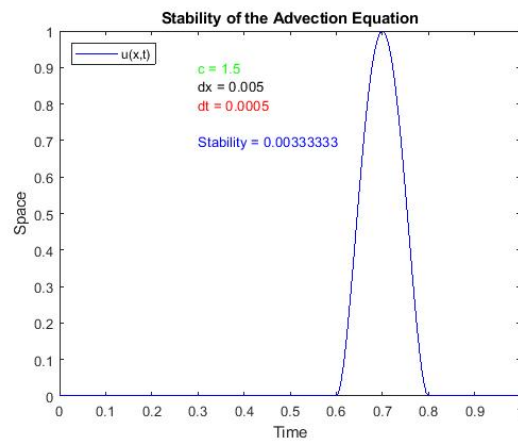


Figure 15: Case-3 for Stability

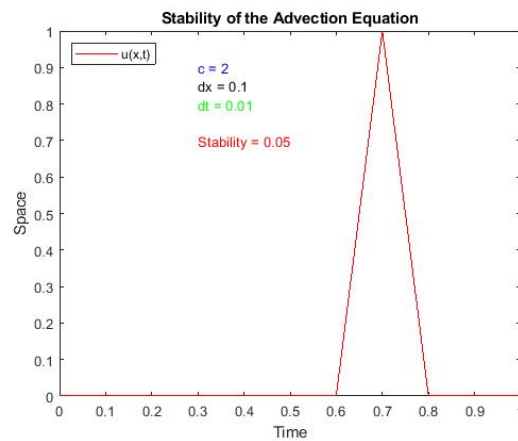


Figure 16: Case-4 For Stability

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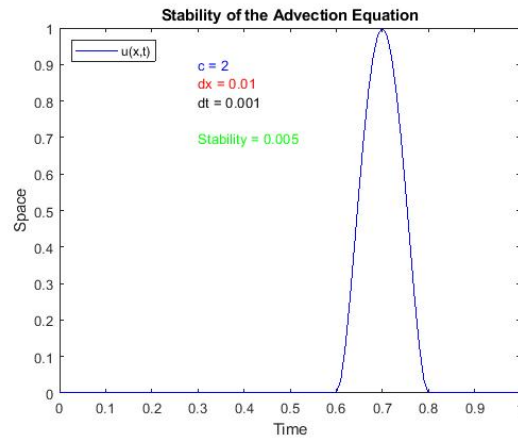


Figure 17: Case-4 For Stability

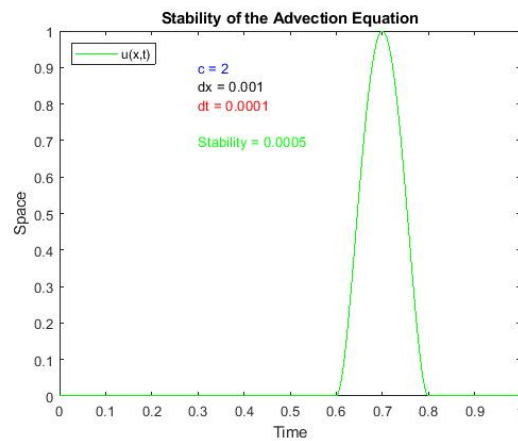


Figure 18: Case-4 For Stability

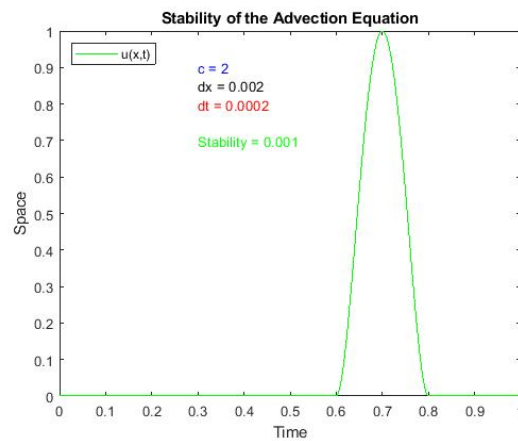


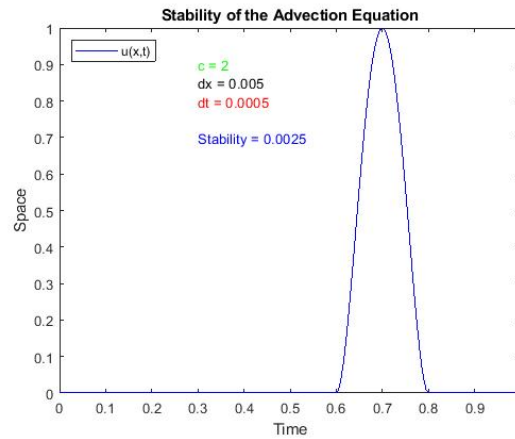
Figure 19: Case-4 for Stability

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<http://amresearchreview.com/index.php/Journal/about>*Figure 20: Case-4 for Stability*

4. DISCUSSION

In this paper, we have given a numerical solution of the advection equation periodic using the FTCS (forward-time centered-space) method, with the help of periodic conditions for boundary. The advection equation describes the transport of a quantity by a fluid, and has a wide range of applications in physics, engineering, and other fields. The FTCS method is a simple and widely used method for solving partial differential equations and is particularly suited to problems with advection-like behavior.

The numerical scheme used in this paper involves discretizing the advection equation in space and time using finite differences. These spatial derivatives can be estimated by centered differences, and the time derivatives by forward differences. The resulting numerical scheme is explicit, meaning that the solution at each time step can be computed directly from the previous time step without the need for matrix inversion or other iterative methods.

We tested the different sizes for stability of the FTCS method using a simple example problem in introduction and found that the method produced stability results for a range of spatial and temporal resolutions. However, we also found that the method is conditionally firm, meaning that the time step size must be opted carefully to ensure stability. Specifically, the time step size must be less than a certain value that depends on the spatial resolution and the advection velocity. If the

time step size is too large, the method becomes unstable and produces unphysical oscillations in the solution.

One limitation of the FTCS method is that it is only first-order accurate in time, meaning that the error in the solution scales linearly with the time step size. This can be a disadvantage in problems where high temporal accuracy is required. Additionally, the method is only accurate for advection-dominated problems and may produce inaccurate results in problems where diffusion or other effects are significant.

Overall, the FTCS method stipulates a simple and effectual approach to solving advection problems and can be a valuable tool for studying the transport of varieties of volume in fluids. However, precautions must be acquired to choose suitable step sizes of time and to authenticate the accuracy and stability of the schemes for every specific problem under the consideration. Future work could investigate the use of higher-order time integration techniques or alternative schemes that numerically improve the stability and accuracy of the solution.

5. CONCLUSION

In this particular research paper, we have offered a numerical method for solving the advection equation using FTCS method with periodic boundary constraints. We have exhibited that the method or scheme is applicable for computing the solution of the advection equation but demands careful selection of the spatial and time steps to guarantee stability. Our results implies that the FTCS method can be a useful tool for investigating the behaviors of fluids in diverse applications.

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