

Explore Ghayasudin Fractional Operators and Improve Some Inequalities through Convexity

Saqib Hayat khan

Department of Biological Sciences Superior University, 10 KM Lahore- Sargodha Rd, Sargodha, Punjab 40100, Pakistan. Saqib8371198@gmail.com

Faisal Khan

Department of Biological Sciences Superior University, 10 KM Lahore- Sargodha Rd, Sargodha, Punjab 40100, Pakistan. faisalkhan190897@gmail.com

Aqib Mushtaq Khan

Department of Biological Sciences Superior University, 10 KM Lahore- Sargodha Rd, Sargodha, Punjab 40100, Pakistan. aqibmushtaqkhan333@gmail.com

Ali Ahmed

Department of Biological Sciences Superior University, 10 KM Lahore- Sargodha Rd, Sargodha, Punjab 40100, Pakistan. m.ahmad.sgd@superior.edu.pk

ABSTRACT

Special functions play a crucial role in the progression from classical calculus to fractional calculus, with extensive applications across all fields of pure and applied mathematics. In this study, we discuss the different versions of Bessel-Maitland functions and its generalizations, fractional operators having multi-index Bessel-Maitland functions as its kernel. The modification of well-known inequalities like as Hermite-Hadamard (HH), Fejér type inequalities with refinements are also considered by implementation of generalized fractional operators named as Ghayasuddin fractional. We also examine the interrelationships among these well-known inequalities (HH, TI, FI, and MPI). Furthermore, we extend these integral inequalities to the class of (η_1, η_2) -convex functions and derive additional related inequalities as consequences of our main results. Moreover, we discuss the fractional and differential behavior of generalized version of Bessel-Maitland function of six parameters.

Keywords: Fractional calculus, Bessel-Maitland functions, Generalized fractional operators, Ghayasuddin fractional operators, Hermite-Hadamard inequalities, Fejér type inequalities

Introduction

Special functions occupy a fundamental position in the domain of applied mathematics, serving as indispensable tools for modeling, analysis, and problem-solving across diverse branches of science and engineering. Their systematic development, which can be traced back to the 19th century, was largely stimulated by pioneering contributions of eminent Swiss mathematicians. Among the classical special functions, the Beta and Gamma functions stand as archetypes, being expressible not only in terms of infinite series but also in integral formulations. These functions, along with others, are often introduced via generating functions and are intricately linked with differential and integral equations that encapsulate their structural properties. The theoretical significance and practical versatility of special functions have been documented extensively in monographs such as *Special Functions* [1]. Beyond their historical foundations, special functions have demonstrated remarkable utility in both pure and applied mathematical contexts. Functions such as the Beta, Gamma, and hypergeometric functions have resolved numerous analytical challenges in the natural sciences and engineering. Frequently, these functions emerge as closed-form solutions to differential equations or as integrals of elementary functions. Their innovations in theoretical constructs and applied techniques have profoundly influenced scientific progress. In recent decades, special functions, in conjunction with fractional operators, have proven particularly effective in addressing fractional-order differential equations, thereby extending classical analysis to phenomena that cannot be expressed adequately by conventional integral formulations [2,3]. Consequently, the field of fractional calculus has garnered significant attention, fostering new mathematical methodologies and practical applications. The last few decades have witnessed an exponential increase in the definition and generalization of novel fractional operators [4,5]. The conceptual origins of fractional calculus can be traced to the hypotheses of Leibniz (1695, 1697) and Euler (1730), whose groundbreaking work laid the theoretical foundation for this discipline. Euler's discovery of the Gamma function further consolidated the central role of special functions in fractional calculus, numerical analysis, and functional analysis [1,6]. Contemporary research has extended this tradition by deriving a substantial body of fractional integral identities involving new and modified versions of classical functions, which are increasingly applied to real-world problems in dynamic and impactful ways.

Fractional calculus, as a branch of mathematical analysis, generalizes the notion of integral and differential operators to arbitrary (real or complex) orders. This extension has transformed the treatment of integral and differential equations by allowing more flexible representations of phenomena in mathematical physics. From the contributions of Laplace, Fourier, Abel, Liouville, Riemann, and Weyl, to the systematic work of Erdélyi, the intellectual trajectory of fractional calculus reflects a rich heritage of innovation. Its formal establishment as a distinct field was further consolidated by the seminal text of Oldham and Spanier, which provided the first comprehensive treatment of the subject. Today, fractional calculus constitutes a rapidly advancing research frontier, with profound implications for both theoretical mathematics and applied sciences. A classical yet continually evolving special function is the Bessel function, first introduced by Daniel Bernoulli in 1732 through a series solution describing the oscillatory displacement of a heavy hanging chain. This foundational result was extended by Euler in 1764, who derived a related series representation. The German astronomer Friedrich Wilhelm Bessel provided a systematic study in 1824, formalizing the general properties of these functions. Subsequently, Bessel functions became central to mathematical physics, as highlighted in Watson's monumental 1922 work *A Treatise on the Theory of Bessel Functions*. Their applications span diverse areas such as electrical engineering, wave propagation, diffraction theory, and hydrodynamics [7]. Parallel to the development of special functions, convex analysis has emerged as another cornerstone of applied and computational mathematics. Convex functions and convex sets are valued for their controllability, robustness, and mathematical elegance. The generalizations of convexity, such as invexity and preinvexity, introduced by Hanson and others, have broadened the theoretical scope of inequality analysis. The Hermite–Hadamard inequality, one of the most extensively studied integral inequalities based on convexity, exemplifies the profound implications of these generalizations. Such inequalities, with their numerous refinements and extensions, remain indispensable in mathematical analysis and optimization theory.

Literature Support

The centrality of Bessel functions to mathematics is underscored by their wide-ranging applications in physics, engineering, and applied sciences. They can be expressed and analyzed through both differential and integral formulations, yielding solutions of

remarkable generality [8–10]. Over time, modifications of Bessel functions incorporating additional parameters have been developed, thereby giving rise to extended classes of Bessel-type functions [11–17]. These generalizations have enabled more flexible analytical frameworks and have produced significant results in diverse domains of mathematics [18,19].

One notable generalization, the Bessel–Maitland function, was introduced by Edward Maitland Wright via a series representation:

$$J_{\beta}^{\alpha}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(\alpha n + \beta + 1)},$$

which has since been applied in mathematical engineering, physics, chemistry, and biology [20]. Subsequent extensions include the generalization introduced by Pathak (1966), which has been widely studied for its properties and analytical applications [21]. Further work has produced the extended Wright–Bessel function (also referred to as the extended Bessel–Maitland function), whose structural properties and associated integral transforms including Mellin transforms have been systematically explored [22,23]. A wide range of integrals involving special functions has also been analyzed, yielding powerful results with direct applications in applied analysis [24–28]. In 1971, Prabhakar introduced an integral operator with the generalized Mittag–Leffler function as its kernel [29]. Since then, this kernel has become a standard element of fractional integral operators, further enhancing the depth of fractional calculus [30,31]. Similarly, the Bessel–Maitland function has been utilized as a kernel in integral operators [32], enriching both theoretical development and computational methodologies. And a variety of integral operators based on k-functions have been studied in detail, offering new perspectives in functional analysis [33–38].

Preliminaries

In this we discuss fundamental definitions related to special functions, fractional operators, and their generalizations. These concepts serve as the foundation for deriving our main results.

Gamma Function:	The gamma function [39] is defined for $\Re(z) > 0$ via the integral representation:	$\Gamma(z) = \int_0^\infty g^{z-1} e^{-g} dg.$
Beta function	Beta function defined for $\Re(x) > 0$, $\Re(z) > 0$ in integral representation and also with gamma functions relations can be write as:	$B(x, z) = \int_0^1 g^{x-1} (1-g)^{z-1} dg = \frac{\Gamma(x)\Gamma(z)}{\Gamma(x+z)}$
The classical Gauss hypergeometric function	The Gauss hypergeometric function, as defined by Beukers [39], is valid for $ z < 1$ with parameters $\xi, \zeta, \delta \in \mathbb{C}$ and $\delta \neq 0$, where $(\xi)_n$, $(\zeta)_n$, and $(\delta)_n$ denote the Pochhammer symbols, and is given by:	${}_2F_1(\xi, \zeta, \delta, z) = \sum_{n=0}^{\infty} \frac{(\xi)_n (\zeta)_n z^n}{(\delta)_n n!}.$ When $z=1$ then the Gauss hypergeometric function in the form of gamma function can be defined for $\Re(\delta - \xi - \zeta) > 0$ as ${}_2F_1(\xi, \zeta, \delta, 1) = \frac{\Gamma(\delta)\Gamma(\delta - \xi - \zeta)}{\Gamma(\delta - \xi)\Gamma(\delta - \zeta)}.$
Dirichlet formula	Dirichlet formula (Fubini's theorem), [24] defined as	$\int_x^y du \int_x^u z(u, v) dv = \int_x^y dv \int_v^y z(u, v) du.$
Lebesgue measurable space	The Lebesgue measurable space [40] is defined for $c > a$ as follows:	$\mathbb{L}(a, c) = \{g(z) :$
Bessel-Maitland function	Edward Maitland Wright [41] introduced a function known as the Fox-Wright function, which generalizes the Bessel function. The Bessel-Maitland function is represented as:	$J_{\beta}^{\alpha}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(\alpha n + \beta + 1)} = \phi(\alpha, \beta + 1; -z),$
Generalization of Bessel-Maitland function in four parameters defined by		$J_{\beta, q}^{\alpha, \gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} (-z)^n}{n! \Gamma(\alpha n + \beta + 1)},$

Bessel- [21] for $\Re(\alpha) > 0, \Re(\beta) \geq -1, \Re(\gamma) > 0$ where $\alpha, \beta, \gamma \in \mathbb{C}$.

Maitland and $q \in (0,1) \cup \mathbb{N}$ as
function

$$J_{\beta, \gamma, \delta}^{\alpha, b, d}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{bn} (-z)^n}{\Gamma(\alpha n + \beta + 1)(\delta)_{dn}}.$$

An The extended Bessel-Maitland function,
extension which serves as a further generalization of
of the the Bessel-Maitland function [22], is
Bessel- defined for $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\Re(\alpha) >$
Maitland $0, \Re(\beta) \geq -1, \Re(\gamma) > 0, \Re(\delta) > 0$. Also, let
function $b, d \in \mathbb{R}^+$ with $b < \Re(\alpha) + d$ represents as

Throughout this work, we use the notation $\mathbb{C}$ for the set of complex numbers, $\mathbb{R}$ for the set of positive real numbers, and $\mathbb{N}$ for the set of positive integers. The function introduced herein is a generalized form of the Bessel-Maitland function, defined with six parameters.

Generaliz
ed Bessel-
Maitland
function-

The generalized Bessel-Maitland function [42], is elucidate as follows

$$\mathbb{J}_{\nu, \eta_1, \rho_1, \gamma_1}^{\mu, \xi, m, \sigma}(s) = \sum_{n=0}^{\infty} \frac{(\eta_1)_{\xi n} (\gamma_1)_{\sigma n} (-s)^n}{\Gamma(\beta n + \alpha + 1)(\rho_1)_{mn}}.$$

III

Following the above-mentioned equations, our study also incorporated an in-depth focus on several advanced mathematical constructs and operators. These include the Bessel-Maitland function-IV, Generalized Srivastava fractional integral operator, Ghayasuddin fractional integral operator, Extension of generalized Ghayasuddin fractional integral operator, New version of generalized Ghayasuddin fractional integral operator, Bessel-Maitland k-function, Generalized Bessel-Maitland k-function, Extended Bessel-Maitland k-function, Srivastava fractional integral k-operator, Generalized Srivastava fractional integral k-operator, Ghayasuddin fractional integral k-operator ensuring a comprehensive treatment of both classical and k-generalized formulations within the fractional calculus framework.

Explore Some Inequalities by Ghayasuddin Fractional Operators

In this we discuss the new version of fractional operators; named as Ghayasuddin fractional operators, and improved some fractional inequalities like as Hermite-Hadamard

inequality. Fejér type inequalities by different convexities. This section presents Hermite-Hadamard inequalities and their refinements, obtained through the application of Ghayasuddin fractional operators that utilize an eight-parameter generalized Bessel-Maitland function as their kernel, along with convex functions.

Theorem 4.1.1. Let $\rho : [u_1, u_2] \rightarrow \mathbb{R}$ be an increasing and convex function on $[u_1, u_2]$, such that $0 < u_1 < u_2$ and $\rho \in L_1[u_1, u_2]$. Then, for the generalized fractional integrals, we have:

$$\begin{aligned} \rho\left(\frac{u_1 + u_2}{2}\right) \left(\mathcal{J}_{\beta, \mu, \pi, w; v}^{\alpha, v, p} 1\right)(u_1) &\leq \frac{1}{2} \left[(\mathcal{J}_{\beta, \mu, \pi, w; v}^{\alpha, v, p} \rho)(u_1) + (\mathcal{J}_{\beta, \mu, \pi, w; u}^{\alpha, v, p} \rho)(u_2) \right] \\ &\leq \frac{\rho(u_1) + \rho(u_2)}{2} \left(\mathcal{J}_{\beta, \mu, \pi, w; u}^{\alpha, v, p} 1\right)(u_2). \end{aligned}$$

where $\alpha, \beta, \mu, \pi, w \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) \geq -1$, $\Re(\mu) > 0$, $\Re(\pi) > 0$, $\Re(\gamma) > 0$; $v, p, q \geq 0$, and $v, p > \Re(\alpha) + q$

Proof. By the convexity of ρ on the interval $[u_1, u_2]$, let $s, y \in [u_1, u_2]$ with $\rho = \frac{1}{2}$, we have

$$\rho\left(\frac{s+y}{2}\right) \leq \frac{\rho(s) + \rho(y)}{2},$$

considering

$$s = \rho u_1 + (1 - \rho) u_2, \quad y = (1 - \rho) u_1 + \rho u_2$$

we obtain

$$2\rho\left(\frac{u_1+u_2}{2}\right) \leq \rho(\rho u_1 + (1-\rho)u_2) + \rho((1-\rho)u_1 + \rho u_2).$$

Multiplication of $(1-\rho)^\beta (\mathbb{J}_{\beta,\mu,\pi}^{\alpha,v,p})(w(1-\rho)^\alpha)$ on both sides and by integration of the inequality on $[0, 1]$ with respect to ρ , we achieve

$$\begin{aligned} & 2\rho\left(\frac{u_1+u_2}{2}\right) \int_0^1 (1-\rho)^\beta \mathbb{J}_{\beta,\mu,\pi}^{\alpha,v,p}(w(1-\rho)^\alpha) d\rho \\ & \leq \int_0^1 (1-\rho)^\beta \mathbb{J}_{\beta,\mu,\pi}^{\alpha,v,p}(w(1-\rho)^\alpha) \rho(\rho u_1 + (1-\rho)u_2) d\rho \\ & \quad + \int_0^1 (1-\rho)^\beta \mathbb{J}_{\beta,\mu,\pi}^{\alpha,v,p}(w(1-\rho)^\alpha) \rho((1-\rho)u_1 + \rho u_2) d\rho. \\ & 2\rho\left(\frac{u_1+u_2}{2}\right) \sum_{n=0}^{\infty} \frac{(\mu)_{vn}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \int_0^1 (1-\rho)^{\beta+\alpha n} d\rho \\ & \leq \sum_{n=0}^{\infty} \frac{(\mu)_{vn}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \times \\ & \quad \left[\int_0^1 (1-\rho)^{\beta+\alpha n} \rho(\rho u_1 + (1-\rho)u_2) d\rho + \int_0^1 (1-\rho)^{\beta+\alpha n} \rho((1-\rho)u_1 + \rho u_2) d\rho \right]. \end{aligned} \quad (4.1)$$

Using substitution in (5.1) results as,

$$\begin{aligned} & 2\rho\left(\frac{u_1+u_2}{2}\right) (\mathcal{I}_{\beta,\mu,\pi,w;u_2}^{\alpha,v,p} 1)(u_1) \leq \left[\int_{u_1}^{u_2} \left(\frac{z-u_1}{u_2-u_1}\right)^\beta \mathbb{J}_{\beta,\mu,\pi}^{\alpha,v,p}\left(w\left(\frac{z-u_1}{u_2-u_1}\right)^\alpha\right) \rho(z) dz \right. \\ & \quad \left. + \int_{u_1}^{u_2} \left(\frac{u_2-z}{u_2-u_1}\right)^\beta (\mathbb{J}_{\beta,\mu,\pi}^{\alpha,v,p})\left(w\left(\frac{u_2-z}{u_2-u_1}\right)^\alpha\right) \rho(z) dz \right] \\ & \rho\left(\frac{u_1+u_2}{2}\right) (\mathcal{I}_{\beta,\mu,\pi,w;v-1}^{\alpha,v,p})(u_1) \leq \frac{1}{2} \left[(\mathcal{I}_{\beta,\mu,\pi,w;u_2}^{\alpha,v,p} \rho)(u_1) + (\mathcal{I}_{\beta,\mu,\pi,\gamma,w;u_1}^{\alpha,v,p,q} \rho)(u_2) \right] \end{aligned} \quad (4.2)$$

From the convexity of ρ , proceeding to next part of inequality, by using:

$$\rho(\rho u_1 + (1 - \rho)u_2) \leq \rho\rho(u_1) + (1 - \rho)\rho(u_2)$$

$$\rho((1 - \rho)u_1 + \rho u_2) \leq (1 - \rho)\rho(u_1) + \rho\rho(u_2)$$

On adding the above inequalities we obtain

$$\rho(\rho u_1 + (1 - \rho)u_2) + \rho((1 - \rho)u_1 + \rho u_2) \leq (\rho(u_1) + \rho(u_2))$$

Multiplication of $(1 - \rho)^\beta \mathbb{J}_{\beta, \mu, \pi}^{\alpha, v, p}(w(1 - \rho)^\alpha)$ on both sides of inequality and with respect to ρ integration of the resulting inequality on $[0, 1]$, we obtain

$$\begin{aligned} & \int_0^1 (1 - \rho)^\beta \mathbb{J}_{\beta, \mu, \pi}^{\alpha, v, p}(w(1 - \rho)^\alpha) \rho(\rho u_1 + (1 - \rho)u_2) d\rho \\ & + \int_0^1 (1 - \rho)^\beta \mathbb{J}_{\beta, \mu, \pi}^{\alpha, v, p}(w(1 - \rho)^\alpha) \rho((1 - \rho)u_1 + \rho u_2) d\rho \\ & \leq (\rho(u_1) + \rho(u_2)) \int_0^1 (1 - \rho)^\beta \mathbb{J}_{\beta, \mu, \pi}^{\alpha, v, p}(w(1 - \rho)^\alpha) d\rho. \end{aligned}$$

By substitution in the integrals involved gives,

$$\frac{1}{2} \left[(\mathbb{J}_{\beta, \mu, \pi, w; u_2^-}^{\alpha, v, p} \rho)(u_1) + (\mathbb{J}_{\beta, \mu, \pi, w; u_1^+}^{\alpha, v, p, q} \rho)(u_2) \right] \leq \frac{\rho(u_1) + \rho(u_2)}{2} \left(\mathbb{J}_{\beta, \mu, \pi, w; u_1^+}^{\alpha, v, p} 1 \right)(u_2) \quad (4.3)$$

combining the results, we have the desired inequality. $\square$

New Version Trapezoidal Inequalities Attributed To Hermite-Hadamard Inequalities

Lemma 4.2.1. Let $\alpha, \beta, \mu, \pi, \gamma, w \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) \geq -1$, $\Re(\mu) > 0$, $\Re(\pi) > 0$; $v, p, q, w \geq 0$, and $v, p > \Re(\alpha) + q$, consider a function $\rho : B \rightarrow \mathbb{R}$ such that $B = [u_1, u_2] \subseteq \mathbb{R}$, $\rho' \in$

$L_1[u_1, u_2]$ be a differentiable function on (u_1, u_2) . Then we get

$$\begin{aligned} & \frac{\rho(u_1) + \rho(u_2)}{2} \mathbb{J}_{\beta, \mu, \pi, \gamma}^{\alpha, \nu, p, q}(w(1)^\alpha) - \frac{1}{2(u_2 - u_1)} \left[\mathfrak{J}_{(\beta-1), \mu, \pi, \gamma, w; u_1}^{\alpha, \nu, p, q} \rho(u_2) + \mathfrak{J}_{(\beta-1), \mu, \pi, \gamma, w; u_2}^{\alpha, \nu, p, q} \rho(u_1) \right] \\ &= \frac{u_2 - u_1}{2} G \end{aligned}$$

for

$$\begin{aligned} G &= \int_0^1 (1-\rho)^\beta \mathbb{J}_{\beta, \mu, \pi}^{\alpha, \nu, p}(w(1-\rho)^\alpha) \rho'(\rho u_1 + (1-\rho)u_2) d\rho + \\ & \int_0^1 (-\rho)^\beta \mathbb{J}_{\beta, \mu, \pi}^{\alpha, \nu, p}(w(\rho)^\alpha) \rho'(\rho u_1 + (1-\rho)u_2) d\rho \end{aligned}$$

Utilizing the lemma, we present the following theorem.

Theorem 4.2.1. Let a function $\rho : B = [u_1, u_2] \rightarrow \mathbb{R}$ be differentiable on the open interval (u_1, u_2) . Assume that $|\rho'|$ is convex on B . Then, for the generalized fractional integrals defined in (3.12.1), we have:

$$\begin{aligned} & \left| \frac{\rho(u_1) + \rho(u_2)}{2} \mathbb{J}_{\beta, \mu, \pi, \gamma}^{\alpha, \nu, p}(w(1)^\alpha) - \frac{1}{2(u_2 - u_1)} \times \left[\mathfrak{J}_{(\beta-1), \mu, \pi, \gamma, w; u_1}^{\alpha, \nu, p, q} \rho(u_2) + \mathfrak{J}_{(\beta-1), \mu, \pi, \gamma, w; u_2}^{\alpha, \nu, p, q} \rho(u_1) \right] \right| \\ & \leq \frac{(u_2 - u_1)}{2} \left| \mathbb{J}_{(\beta+1), \mu, \pi}^{\alpha, \nu, p}(w(1)^\alpha) - \frac{1}{2^\beta} \mathbb{J}_{(\beta+1), \mu, \pi}^{\alpha, \nu, p}(w(\frac{1}{2})^\alpha) \right| \left[|\rho'(u_1)| + |\rho'(u_2)| \right]. \end{aligned}$$

where $\beta' \geq 0, \alpha, \beta, \mu, \pi, w \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) \geq -1, \Re(\mu) > 0, \Re(\pi) > 0, \Re(\gamma) > 0$; $\nu, p, q, w \geq 0$, and $\nu, p > \Re(\alpha) + q$.

Proof. The following integral expression is taken into consideration

$$\begin{aligned}
 & \left| \frac{\rho(u_1) + \rho(u_2)}{2} \mathbb{J}_{\beta, \mu, \pi}^{\alpha, \nu, p}(w(1)^\alpha) - \frac{1}{2(u_2 - u_1)} \right. \\
 & \times \left[\mathcal{I}_{(\beta-1), \mu, \pi, w; u_1}^{\alpha, \nu, p} \rho(u_2) + \mathcal{I}_{(\beta-1), \mu, \pi, w; u_2}^{\alpha, \nu, p} \rho(u_1) \right] \Big| \\
 & = \left| \left(\frac{u_2 - u_1}{2} \right) \mathbb{I} \right| \\
 & \leq \left(\frac{u_2 - u_1}{2} \right) \sum_{n=0}^{\infty} \left| \frac{(\mu)_{\nu n}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \right| \times \int_0^1 \left| (1 - \rho)^{\beta + \alpha n} - \rho^{\beta + \alpha n} \right| \\
 & \left| \rho'(u_1 \rho + (1 - \rho)u_2) \right| d\rho \\
 & \leq \left(\frac{u_2 - u_1}{2} \right) \sum_{n=0}^{\infty} \left| \frac{(\mu)_{\nu n}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \right| \times \\
 & \int_0^1 \left| (1 - \rho)^{\beta + \alpha n} - \rho^{\beta + \alpha n} \right| \left[|\rho'(u_1)| \rho + (1 - \rho) |\rho'(u_2)| \right] d\rho \\
 & = \left(\frac{u_2 - u_1}{2} \right) \sum_{n=0}^{\infty} \left| \frac{(\mu)_{\nu n}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \right| \\
 & \times \int_0^{\frac{1}{2}} \left((1 - \rho)^{\beta + \alpha n} - \rho^{\beta + \alpha n} \right) \left[|\rho'(u_1)| \rho + (1 - \rho) |\rho'(u_2)| \right] d\rho \\
 & + \int_{\frac{1}{2}}^1 \left(\rho^{\beta + \alpha n} - (1 - \rho)^{\beta + \alpha n} \right) \left[|\rho'(u_1)| \rho + (1 - \rho) |\rho'(u_2)| \right] d\rho.
 \end{aligned}$$

Solving the integrals involved by using integration by parts method, we obtain

$$\begin{aligned}
 & \left| \frac{\rho(u_1) + \rho(u_2)}{2} \mathbb{J}_{\beta, \mu, \pi}^{\alpha, \nu, p}(w(1)^\alpha) - \frac{1}{2(u_2 - u_1)} \right. \\
 & \times \left[\mathcal{I}_{(\beta-1), \mu, \pi, w; u_1}^{\alpha, \nu, p} \rho(u_2) + \mathcal{I}_{(\beta-1), \mu, \pi, w; u_2}^{\alpha, \nu, p} \rho(u_1) \right] \Big| \\
 & \leq \frac{(u_2 - u_1)}{2} \left| \mathbb{J}_{(\beta+1), \mu, \pi}^{\alpha, \nu, p}(w(1)^\alpha) - \frac{1}{2\beta} \mathbb{J}_{(\beta+1), \mu, \pi}^{\alpha, \nu, p}(w(\frac{1}{2})^\alpha) \right| \left[|\rho'(u_1)| + |\rho'(u_2)| \right].
 \end{aligned}$$

Fejér and Fejér-Hadamard Kind of Inequalities having (η_1, η_2) -Convex Function

This section mentioned Fejér type inequalities for a function to be (η_1, η_2) -convex by utilizing the generalized fractional integral operator having generalized Bessel-Maitland-17 III as a kernel in definition (3.12.1) as follows.

Theorem 4.3.1. Consider for an (η_1, η_2) -convex function such that η_2 is an integrable bi-function, suppose $\rho : B \rightarrow \mathbb{R}$, and $\rho(B) \times \rho(B)$ then for $u_1, u_2 \in B$, $\eta_1(u_2, u_1)$ having $\rho \in L_1[u_1, u_1 + \eta_1(u_2, u_1)]$ and the function θ is integrable and symmetric to $u_1 + \frac{1}{2}\eta_1(u_2, u_1)$ i.e., $\theta(2u_1 + \eta_1(u_2, u_1) - s) = \theta(s)$, for: $[u_1, u_1 + \eta_1(u_2, u_1)] \rightarrow \mathbb{R}^+$ where $B \subseteq \mathbb{R}$ be an invex set with respect to η_1 such that

$$\eta_1(u_2 + \rho_2 \eta_1(u_1, u_2), u_2 + \rho_1 \eta_1(u_1, u_2)) = (\rho_2 - \rho_1) \eta_1(u_1, u_2) \quad (4.4)$$

for all $\rho_1, \rho_2 \in [0, 1]$. Based on the generalized fractional integrals defined in (3.12.1), the following Fejér-type inequality holds:

$$\begin{aligned} & \rho \left(\frac{2u_1 + \eta_1(u_2, u_1)}{2} \right) \left[(\mathcal{J}_{\beta, u, \eta, w; (u_1 + \eta_1(u_2, u_1))^-}^{\alpha, v, p} \theta)(u_1) + (\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \theta)(u_1 + \eta_1(u_2, u_1)) \right] \\ & - \frac{1}{2} \int_{u_1}^{u_1 + \eta_1(u_2, u_1)} \left[(u_1 + \eta_1(u_2, u_1) - s)^{\beta} \mathbb{J}_{\beta, \mu, \pi}^{\alpha, v, p}(w(u_1 + \eta_1(u_2, u_1) - s))^{\alpha} \right. \\ & \left. + (s - u_1)^{\beta} \mathbb{J}_{\beta, \mu, \pi}^{\alpha, v, p}(w(s - u_1))^{\alpha} \right] \times \eta_2(\rho(s), \rho(2u_1 + \eta_1(u_2, u_1) - s)) \theta(s) ds \\ & \leq \left[(\mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1))^-}^{\alpha, v, p} \rho \theta)(u_1) + (\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \rho \theta)(u_1 + \eta_1(u_2, u_1)) \right]. \end{aligned}$$

where $\alpha, \beta, \mu, \pi, \gamma, w \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) \geq -1$, $\Re(\mu) > 0$, $\Re(\pi) > 0$; $v, p, q, \geq 0$, and $v, p > \Re(\alpha) + q$.

Proof. According to given condition utilizing (η_1, η_2) -convexity of the function ρ and the equation (5.4) obtained result will be

$$\begin{aligned} & \rho \left(\frac{2u_1 + \eta_1(u_2, u_1)}{2} \right) = \rho \left(\frac{2u_1 + (1 + \rho) \eta_1(u_2, u_1)}{2} - \frac{\rho \eta_1(u_2, u_1)}{2} \right) \\ & = \rho \left(\frac{2u_1 + (1 + \rho) \eta_1(u_2, u_1)}{2} + \frac{1}{2} \eta_1 \left(u_1 + \frac{(1 - \rho)}{2} \eta_1(u_2, u_1), u_1 + \frac{(1 + \rho)}{2} \eta_1(u_2, u_1) \right) \right), \end{aligned}$$

$$\begin{aligned} & \rho \left(\frac{2u_1 + \eta_1(u_2, u_1)}{2} \right) \\ & \leq \rho \left(\frac{2u_1 + (1+\rho)\eta_1(u_2, u_1)}{2} \right) + \frac{1}{2}\eta_2 \left(\rho \left(\frac{2u_1 + (1-\rho)\eta_1(u_2, u_1)}{2} \right), \rho \left(\frac{2u_1 + (1+\rho)\eta_1(u_2, u_1)}{2} \right) \right) \end{aligned} \quad (4.5)$$

Adopting the same course of action as above, we get

$$\begin{aligned} & \rho \left(\frac{2u_1 + \eta_1(u_2, u_1)}{2} \right) \\ & \leq \rho \left(\frac{2u_1 + (1-\rho)\eta_1(u_2, u_1)}{2} \right) + \frac{1}{2}\eta_2 \left(\rho \left(\frac{2u_1 + (1+\rho)\eta_1(u_2, u_1)}{2} \right), \rho \left(\frac{2u_1 + (1-\rho)\eta_1(u_2, u_1)}{2} \right) \right) \end{aligned} \quad (4.6)$$

Utilization of the generalized fractional operator defined in definition(3.12.1) for the integral we have

$$\begin{aligned} I_1 &= \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1))^-}^{\alpha, v, p} \rho \theta \right) (u_1) \\ &= \int_{u_1}^{(u_1 + \eta_1(u_2, u_1))} (s - u_1)^\beta \mathbb{J}_{\beta, \mu, \pi}^{\alpha, v, p} (w(s - u_1)^\alpha) \rho(s) \theta(s) ds \\ &= \sum_{n=0}^{\infty} \frac{(\mu)_{vn}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \int_{u_1}^{(u_1 + \eta_1(u_2, u_1))} (s - u_1)^{\beta + \alpha n} \rho(s) \theta(s) ds \\ I_1 &= \sum_{n=0}^{\infty} \frac{(\mu)_{vn}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \left[\int_{u_1}^{u_1 + \frac{1}{2}\eta_1(u_2, u_1)} (s - u_1)^{\beta + \alpha n} \rho(s) \theta(s) ds \right. \\ & \quad \left. + \int_{u_1 + \frac{1}{2}\eta_1(u_2, u_1)}^{u_1 + \eta_1(u_2, u_1)} (s - u_1)^{\beta + \alpha n} \rho(s) \theta(s) ds \right]. \end{aligned} \quad (4.7)$$

Integrals of above equation on substituting $s = \frac{2u_1 + (1-\rho)\eta_1(u_2, u_1)}{2}$ and $s = \frac{2u_1 + (1+\rho)\eta_1(u_2, u_1)}{2}$ appearing respectively gives the result as follows

$$\begin{aligned} I_1 &= \sum_{n=0}^{\infty} \frac{(\mu)_{vn}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \left(\frac{\eta_1(u_2, u_1)}{2} \right)^{\beta + \alpha n + 1} \\ & \quad \left[\int_0^1 (1-\rho)^{\beta + \alpha n} \rho \left(\frac{2u_1 + (1-\rho)\eta_1(u_2, u_1)}{2} \right) \theta \left(\frac{2u_1 + (1-\rho)\eta_1(u_2, u_1)}{2} \right) d\rho \right. \\ & \quad \left. + \int_0^1 (1+\rho)^{\beta + \alpha n} \rho \left(\frac{2u_1 + (1+\rho)\eta_1(u_2, u_1)}{2} \right) \theta \left(\frac{2u_1 + (1+\rho)\eta_1(u_2, u_1)}{2} \right) d\rho \right]. \end{aligned} \quad (4.8)$$

Utilization of result of derived inequalities (5.5) and (5.6), leads to

$$\begin{aligned}
 I_1 &\geq \sum_{n=0}^{\infty} \frac{(\mu)_{\nu n}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \left(\frac{\eta_1(u_2, u_1)}{2} \right)^{\beta + \alpha n + 1} \\
 &\left[\int_0^1 (1-\rho)^{\beta + \alpha n} \rho \left(\frac{2u_1 + \eta_1(u_2, u_1)}{2} \right) \theta \left(\frac{2u_1 + (1-\rho)\eta_1(u_2, u_1)}{2} \right) d\rho \right. \\
 &\frac{1}{2} \int_0^1 (1-\rho)^{\beta + \alpha n} \eta_2 \left(\rho \frac{2u_1 + (1+\rho)\eta_1(u_2, u_1)}{2} \right), \rho \left(\frac{2u_1 + (1-\rho)\eta_1(u_2, u_1)}{2} \right) \\
 &\theta \left(\frac{2u_1 + (1-\rho)\eta_1(u_2, u_1)}{2} \right) d\rho \\
 &+ \int_0^1 (1+\rho)^{\beta + \alpha n} \rho \left(\frac{2u_1 + \eta_1(u_2, u_1)}{2} \right) \theta \left(\frac{2u_1 + (1+\rho)\eta_1(u_2, u_1)}{2} \right) d\rho \\
 &\left. - \frac{1}{2} \int_0^1 (1+\rho)^{\beta + \alpha n} \eta_2 \left(\rho \left(\frac{2u_1 + (1-\rho)\eta_1(u_2, u_1)}{2} \right), \rho \left(\frac{2u_1 + (1+\rho)\eta_1(u_2, u_1)}{2} \right) \right) \right. \\
 &\left. \theta \left(\frac{2u_1 + (1+\rho)\eta_1(u_2, u_1)}{2} \right) d\rho \right]
 \end{aligned}$$

on solving we get;

$$\begin{aligned}
 I_1 &= \rho \left(\frac{2u_1 + \eta_1(u_2, u_1)}{2} \right) \left(\frac{\eta_1(u_2, u_1)}{2} \right)^{\beta + \alpha n + 1} \sum_{n=0}^{\infty} \frac{(\mu)_{\nu n}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \\
 &\left[\int_0^1 (1-\rho)^{\beta + \alpha n} \theta \left(\frac{2u_1 + (1-\rho)\eta_1(u_2, u_1)}{2} \right) - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(\mu)_{\nu n}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \times \right. \\
 &\left(\frac{\eta_1(u_2, u_1)}{2} \right)^{\beta + \alpha n + 1} \int_0^1 (1-\rho)^{\beta + \alpha n} \eta_2 \left(\rho \left(\frac{2u_1 + (1+\rho)\eta_1(u_2, u_1)}{2} \right), \right. \\
 &\left. \rho \left(\frac{2u_1 + (1-\rho)\eta_1(u_2, u_1)}{2} \right) \right) \times \theta \left(\frac{2u_1 + (1-\rho)\eta_1(u_2, u_1)}{2} \right) d\rho \\
 &+ \rho \left(\frac{2u_1 + \eta_1(u_2, u_1)}{2} \right) \left(\frac{\eta_1(u_2, u_1)}{2} \right)^{\beta + \alpha n + 1} \sum_{n=0}^{\infty} \frac{(\mu)_{\nu n}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \times \\
 &\int_0^1 (1+\rho)^{\beta + \alpha n} \theta \left(\frac{2u_1 + (1+\rho)\eta_1(u_2, u_1)}{2} \right) d\rho \\
 &\left. - \frac{1}{2} \left(\frac{\eta_1(u_2, u_1)}{2} \right)^{\beta + \alpha n + 1} \sum_{n=0}^{\infty} \frac{(\mu)_{\nu n}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \right. \\
 &\left. \int_0^1 (1+\rho)^{\beta + \alpha n} \eta_2 \left(\rho \left(\frac{2u_1 + (1+\rho)\eta_1(u_2, u_1)}{2} \right), \rho \left(\frac{2u_1 + (1+\rho)\eta_1(u_2, u_1)}{2} \right) \right) \right. \\
 &\left. \theta \left(\frac{2u_1 + (1+\rho)\eta_1(u_2, u_1)}{2} \right) d\rho \right].
 \end{aligned}$$

Using the symmetry of θ to $u_1 + \frac{1}{2}\eta_1(u_2, u_1)$, on simplifying arises to the following

result:

$$I_1 \geq \rho \left(\frac{2u_1 + \eta_1(u_2, u_1)}{2} \right) \sum_{n=0}^{\infty} \frac{(\mu)_{vn}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \int_{u_1}^{u_1 + \eta_1(u_2, u_1)} (s - u_1)^{\beta + \alpha n} \theta(s) ds - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(\mu)_{vn}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \times \int_{u_1}^{u_1 + \eta_1(u_2, u_1)} (u_1 + \eta_1(u_2, u_1) - s)^{\beta + \alpha n} \eta_2 \left(\rho(s), \rho(2u_1 + \eta_1(u_2, u_1) - s) \right) \theta(s) ds.$$

On further simplifications follows to:

$$I_1 \geq \rho \left(\frac{2u_1 + \eta_1(u_2, u_1)}{2} \right) \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1))}^{\alpha, v, p} \theta(u_1) \right) - \frac{1}{2} \int_{u_1}^{u_1 + \eta_1(u_2, u_1)} (u_1 + \eta_1(u_2, u_1) - s)^{\beta} \times \mathbb{J}_{\beta, \mu, \pi}^{\alpha, v, p} (w(u_1 + \eta_1(u_2, u_1) - s)^{\alpha}) \times \eta_2(\rho(s), \rho(2u_1 + \eta_1(u_2, u_1) - s)) \theta(s) ds.$$

Now consider

$$I_2 = \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)}^{\alpha, v, p} \rho \theta \right) (u_1 + \eta_1(u_2, u_1)) = \int_{u_1}^{u_1 + \eta_1(u_2, u_1)} (u_1 + \eta_1(u_2, u_1) - s)^{\beta} \mathbb{J}_{\beta, \mu, \pi, \gamma}^{\alpha, v, p, q} (w(u_1 + \eta_1(u_2, u_1) - s)^{\alpha}) \rho(s) \theta(s) ds.$$

For I_2 integral is solved by using same techniques and on suitable substitutions like I_1 , result will be

$$I_2 \geq \rho \left(\frac{2u_1 + \eta_1(u_2, u_1)}{2} \right) \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)}^{\alpha, v, p} \theta \right) (u_1 + \eta_1(u_2, u_1)) - \frac{1}{2} \int_{u_1}^{u_1 + \eta_1(u_2, u_1)} (s - u_1)^{\beta} \mathbb{J}_{\beta, \mu, \pi}^{\alpha, v, p} (w(s - u_1))^{\alpha} \times \eta_2(\rho(s), \rho(2u_1 + \eta_1(u_2, u_1) - s)) \theta(s) ds$$

Addition of integrals I_1 and I_2 goes to desired inequality. $\square$

Corollary 4.3.1. Suppose for Fejér type inequalities presented in theorem (5.3.1) on placing $\eta_1(u_2, u_1) = u_1 - u_2, \forall u_1, u_2 \in B$ and $\alpha, \beta, \mu, \eta, \gamma, w \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) \geq$

$-1, \Re(\mu) > 0, \Re(\pi) > 0; v, p, q, w \geq 0$, and $v, p > \Re(\alpha) + q$ where ρ to be η -convex:

$$\begin{aligned} & \rho\left(\frac{u_1 + u_2}{2}\right) \left[\left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \theta(u_2) \right) + \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_2)^-}^{\alpha, v, p} \theta(u_1) \right) \right] \\ & - \frac{1}{2} \int_{u_1}^{u_2} \left[(u_2 - s)^{\beta} \mathbb{J}_{\beta, \mu, \pi}^{\alpha, v, p} (w(u_2 - s)^{\alpha}) \right. \\ & \left. + (s - u_1)^{\beta} \mathbb{J}_{\beta, \mu, \pi}^{\alpha, v, p} (w(s - u_1)^{\alpha}) \right] \times \eta_2 \left(\rho(s), \rho(u_1 + u_2 - s), \theta(s) \right) ds \\ & \leq \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_2)^-}^{\alpha, v, p} \rho \theta \right)(u_1) + \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \rho \theta \right)(u_2) \\ & \leq \left(\frac{2\rho(u_1) + \eta_2(\rho(u_2), \rho(u_1))}{2} \right) \left[\left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \theta \right)(u_2) + \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_2)^-}^{\alpha, v, p} \theta \right)(u_1) \right]. \end{aligned}$$

result is Fejér inequality.

Theorem 4.3.2. Let $\eta_1 : B \times B \rightarrow \mathbb{R}$ suppose B be an invex $\subseteq \mathbb{R}$. Let $u_1, u_2 \in B$ full filling $\eta_1(u_2, u_1) \geq 0$ also θ is symmetric about $u + \frac{1}{2}\eta_1(u_2, u_1)$ and integrable so that $:[u_1, u_1 + \eta_1(u_2, u_1)] \rightarrow \mathbb{R}$. Then outcome will be:

$$\begin{aligned} & \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \rho \right)(u_1 + \eta_1(u_2, u_1)) = \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1))^-}^{\alpha, v, p} \rho \right)(u_1) \\ & = \frac{1}{2} \left[\left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \rho \right)(u_1 + \eta_1(u_2, u_1)) + \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1))^-}^{\alpha, v, p} \rho \right)(u_1) \right] \end{aligned}$$

where $\alpha, \beta, \mu, \pi, \gamma, w \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) \geq -1, \Re(\mu) > 0, \Re(\pi) > 0, v, p, q, \geq 0$, and $v, p > \Re(\alpha) + q$.

Proof. Let us take

$$\begin{aligned} & \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \rho \right)(u_1 + \eta_1(u_2, u_1)) \\ & = \int_{u_1}^{u_1 + \eta_1(u_2, u_1)} (u_1 + \eta_1(u_2, u_1) - s)^{\beta} \mathbb{J}_{\beta, \mu, \pi}^{\alpha, v, p} (w(u_1 + \eta_1(u_2, u_1) - s)^{\alpha}) \rho(s) ds. \quad (4.9) \end{aligned}$$

Substitution of $s = 2u_1 + \eta_1(u_2, u_1) - \rho$ in integral equation (6.25) leads to

$$\begin{aligned} & \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \rho \right)(u_1 + \eta_1(u_2, u_1)) = \int_{u_1}^{u_1 + \eta_1(u_2, u_1)} (\rho - u_1)^{\beta} \rho(2u_1 + \eta_1(u_2, u_1) - \rho) d\rho \\ & \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \rho \right)(u_1 + \eta_1(u_2, u_1)) = \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1))^-}^{\alpha, v, p} \rho \right)(u_1) \quad (4.10) \end{aligned}$$

Adding $(\mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1))^-}^{\alpha, v, p} \rho)(u_1)$ on both sides of above equation, gives to the required result. $\square$

Theorem 4.3.3. Let, for η_2 be an integrable bi-function consider $\rho : B \rightarrow \mathbb{R}$, is function that is (η_1, η_2) -convex so that on $\rho(B) \times \rho(B)$. Also for $\theta : [u_1, u_1 + \eta_1(u_2, u_1)] \rightarrow \mathbb{R}^+$ where $u_1, u_2 \in B, \eta_1(u_2, u_1) > 0$ having $\rho \in L_1[u_1, u_1 + \eta_1(u_2, u_1)]$. Function θ is symmetric to $u_1 + \frac{1}{2}\eta_1(u_2, u_1)$ and is integrable also. Due to symmetry we have $\theta(2u_1 + \eta_1(u_2, u_1) - s) = \theta(s)$. Also we take η_1 be invex having $B \subseteq \mathbb{R}$. For the generalized fractional integrals defined in definition(3.12.1) the result of Fejérs inequality holds as:

$$\begin{aligned} & \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1))^-}^{\alpha, v, p} \rho \theta \right)(u_1) + \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \rho \theta \right)(u_1 + \eta_1(u_2, u_1)) \\ & \leq \left(\frac{2\rho(u_1) + \eta_2(\rho(u_2), \rho(u_1))}{2} \right) \\ & \left[\left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1))^-}^{\alpha, v, p} \rho \theta \right)(u_1) + \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \rho \theta \right)(u_1 + \eta_1(u_2, u_1)) \right] \end{aligned}$$

where $\alpha, \beta, \mu, \pi, w \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) \geq -1, \Re(\mu) > 0, \Re(\pi) > 0; v, p, q \geq 0$, and $v, p > \Re(\alpha) + q$.

Proof. Consider the integral,

$$\begin{aligned} & \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1))^-}^{\alpha, v, p} \rho \theta \right)(u_1) = \int_{u_1}^{u_1 + \eta_1(u_2, u_1)} (s - u_1)^\beta \mathcal{J}_{\beta, \mu, \pi}^{\alpha, v, p}(w(s - u_1)^\alpha) \rho(s) \theta(s) ds \\ & = \sum_{n=0}^{\infty} \frac{(\mu)_{vn}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \int_{u_1}^{u_1 + \eta_1(u_2, u_1)} (s - u_1)^\beta \rho(s) \theta(s) ds \end{aligned}$$

Make substitution of $s = u_1 + \rho \eta_1(u_2, u_1)$ moves to the following result

$$\begin{aligned} & \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1))^-}^{\alpha, v, p} \rho \theta \right)(u_1) = \sum_{n=0}^{\infty} \frac{(\mu)_{vn}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \times \\ & (\eta_1(u_2, u_1))^{\beta + \alpha n + 1} \int_0^1 (\rho)^{\beta + \alpha n} \rho(u_1 + \rho \eta_1(u_2, u_1)) \theta(u_1 + \rho \eta_1(u_2, u_1)) d\rho. \end{aligned} \quad (4.11)$$

From the given statement as ρ is (η_1, η_2) -convex, we proceed

$$\leq \sum_{n=0}^{\infty} \frac{(\mu)_{vn}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n (\eta_1(u_2, u_1))^{\beta + \alpha n + 1} \int_0^1 (\rho)^{\beta + \alpha n} \left(\rho(u_1) + \rho \eta_2(\rho(u_2), \rho(u_1)) \right) \theta(u_1 + \rho \eta_1(u_2, u_1)) d\rho$$

(4.12)

$$= \sum_{n=0}^{\infty} \frac{(\mu)_{vn}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n (\eta_1(u_2, u_1))^{\beta + \alpha n + 1} \left[\rho(u_1) \int_0^1 (\rho)^{\beta + \alpha n} \theta(u_1 + \rho \eta_1(u_2, u_1)) d\rho + \eta_2(\rho(u_2), \rho(u_1)) \int_0^1 (\rho)^{\beta + \alpha n + 1} \theta(u_1 + \rho \eta_1(u_2, u_1)) d\rho \right]$$

(4.13)

Now, taking the integral

$$\left(\mathfrak{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \rho \theta \right) (u_1 + \eta_1(u_2, u_1)) = \int_{u_1}^{u_1 + \eta_1(u_2, u_1)} (u_1 + \eta_1(u_2, u_1) - s)^{\beta} \mathfrak{J}_{\beta, \mu, \pi}^{\alpha, v, p} (w(u_1 + \eta_1(u_2, u_1) - s)^{\alpha}) \rho(s) \theta(s) ds$$

(4.14)

$$= \sum_{n=0}^{\infty} \frac{(\mu)_{vn}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \times (\eta_1(u_2, u_1))^{\beta + \alpha n + 1} \int_0^1 (\rho)^{\beta + \alpha n} \left(\rho(u_1) + (1 - \rho) \eta_2(\rho(u_2), \rho(u_1)) \right) \theta \left(u_1 + (1 - \rho) \eta_1(u_2, u_1) \right) d\rho$$

$$\leq \sum_{n=0}^{\infty} \frac{(\mu)_{vn}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \times (\eta_1(u_2, u_1))^{\beta + \alpha n + 1} \int_0^1 (\rho)^{\beta + \alpha n} \theta \left(u_1 + (1 - \rho) \eta_1(u_2, u_1) \right) d\rho + \eta_2 \left(\rho(u_2), \rho(u_1) \right) \times \int_0^1 (\rho)^{\beta + \alpha n} (1 - \rho) \theta \left(u_1 + (1 - \rho) \eta_1(u_2, u_1) \right) d\rho$$

(4.15)

By symmetry of θ towards $u_1 + \frac{1}{2}\eta_1(u_2, u_1)$, on addition of equation, we obtain

$$\begin{aligned}
 & (\mathcal{J}_{\beta, \mu, \pi, w; (u_1)+}^{\alpha, v, p} \rho \theta)(u_1 + \eta_1(u_2, u_1)) + (\mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1)) -}^{\alpha, v, p} \rho \theta)(u_1) \\
 & \leq \sum_{n=0}^{\infty} \frac{(\mu)_{vn}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \times (\eta_1(u_2, u_1))^{\beta + \alpha n + 1} \\
 & \left[2\rho(u_1) \int_0^1 (\rho)^{\beta + \alpha n} \theta \left(u_1 + \rho \eta_1(u_2, u_1) \right) d\rho + \eta_2(\rho(u_2), \rho(u_1)) \right. \\
 & \left. \int_0^1 (\rho)^{\beta + \alpha n} \theta \left(u_1 + \rho \eta_1(u_2, u_1) \right) d\rho \right] \\
 & \left(2\rho(u_1 + \eta_2(\rho(u_2), \rho(u_1))) \right) \leq \sum_{n=0}^{\infty} \frac{(\mu)_{vn}}{\Gamma(\alpha n + \beta + 1)(\eta)_{pn}} (-w)^n \times (\eta_1(u_2, u_1))^{\beta + \alpha n + 1} \times \\
 & \int_0^1 (\rho)^{\beta + \alpha n} \theta \left(u_1 + \rho \eta_1(u_2, u_1) \right) d\rho = \left(\frac{2\rho(u_1) + \eta_2(\rho(u_2), \rho(u_1))}{2} \right) \times \\
 & \left(2\mathcal{J}_{\beta, \mu, \pi, \gamma, w; (u_1 + \eta_1(u_2, u_1)) -}^{\alpha, v, p, q} \theta(u_1) \right) \tag{4.16}
 \end{aligned}$$

By using result of the theorem, we obtain

$$\begin{aligned}
 & \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1)) -}^{\alpha, v, p} \rho \theta \right)(u_1) + \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)+}^{\alpha, v, p} \rho \theta \right)(u_1 + \eta_1(u_2, u_1)) \\
 & \leq \left(\frac{2\rho(u_1) + \eta_2(\rho(u_2), \rho(u_1))}{2} \right) \left[(\mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1)) -}^{\alpha, v, p} \rho \theta)(u_1) + \right. \\
 & \left. (\mathcal{J}_{\beta, \mu, \pi, \gamma, w; (u_1)+}^{\alpha, v, p, q} \rho \theta)(u_1 + \eta_1(u_2, u_1)) \right]
 \end{aligned}$$

□

Corollary 4.3.2. Consider for Fejér type inequalities on replacing $\eta_1(u_2, u_1) = u_1 - u_2, \forall u_1, u_2 \in B$ of theorem (5.3.3), for $\alpha, \beta, \mu, \eta, \gamma, w \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) \geq -1, \Re(\mu) >$

0, $\Re(\pi) > 0$; $v, p, q \geq 0$, and $v, p > \Re(\alpha) + q$, we get;

$$\begin{aligned} & \rho\left(\frac{u_1 + u_2}{2}\right) \left[(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \theta(u_2)) + (\mathcal{J}_{\beta, \mu, \pi, w; (u_2)^-}^{\alpha, v, p} \theta(u_1)) \right] \\ & - \frac{1}{2} \int_{u_1}^{u_2} \left[(u_2 - s)^{\beta} \mathbb{J}_{\beta, \mu, \pi}^{\alpha, v, p} (w(u_2 - s)^{\alpha}) \right. \\ & \left. + (s - u_1)^{\beta} \mathbb{J}_{\beta, \mu, \pi}^{\alpha, v, p} (w(s - u_1)^{\alpha}) \right] \times \eta_2(\rho(s), \rho(u_1 + u_2 - s), \theta(s)) ds \\ & \leq \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_2)^-}^{\alpha, v, p} \rho \theta \right)(u_1) \leq \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \rho \theta(u_2) \right) \\ & \leq \left(\frac{2\rho(u_1) + \eta_2(\rho(u_2), \rho(u_1))}{2} \right) \left[(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \theta(u_2)) + (\mathcal{J}_{\beta, \mu, \pi, w; (u_2)^-}^{\alpha, v, p} \theta(u_1)) \right]. \end{aligned}$$

result is Fejér inequality where ρ is be η -convex in generalized fractional integrals.

Corollary 4.3.3. In theorem (5.3.3) on placing $\eta_2(u_1, u_2) = u_1 - u_2$, for all $u_1, u_2 \in \rho(B)$, we achieve

$$\begin{aligned} & \rho\left(\frac{2u_1 + \eta_1(u_2, u_1)}{2}\right) \left[(\mathcal{J}_{\beta, \mu, \pi, \gamma, w; (u_1)^+}^{\alpha, v, p, q} \theta(u_1 + \eta_1(u_2, u_1))) + (\mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1))^-}^{\alpha, v, p} \rho \theta)(u_1 + \eta_1 \right. \\ & \leq \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1))^-}^{\alpha, v, p} \rho \theta \right)(u_1) + \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \theta(u_1 + \eta_1(u_2, u_1)) \right) \\ & \left. \left(\frac{\rho(u_1) + \rho(u_2)}{2} \right) \left[(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \theta(u_1 + \eta_1(u_2, u_1))) + (\mathcal{J}_{\beta, \mu, \pi, \gamma, w; (u_1 + \eta_1(u_2, u_1))^-}^{\alpha, v, p, q} \theta)(u_1) \right] \right] \end{aligned}$$

Result of Fejér Inequality for ρ to be preinvex convex in generalized fractional integral.

Corollary 4.3.4. Consider in function ρ to be convex for Fejér type inequalities (5.3.3) putting $\eta_1(u_1, u_2) = u_1 - u_2, \forall u_1, u_2 \in B$ and $\eta_2(s, y) = s - y, \forall s, y \in \rho(B)$ following result holds

$$\begin{aligned} & \rho\left(\frac{u_1 + u_2}{2}\right) \left[(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \theta(u_1)) + (\mathcal{J}_{\beta, \mu, \pi, w; (u_2)^-}^{\alpha, v, p} \theta(u_1)) \right] \\ & \leq \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_2)^-}^{\alpha, v, p} \rho \theta(u_1) \right) + \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \rho \theta(u_2) \right) \\ & \leq \left(\frac{\rho(u_1) + \rho(u_2)}{2} \right) \left[(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p} \theta(u_2)) + (\mathcal{J}_{\beta, \mu, \pi, w; (u_2)^-}^{\alpha, v, p} \theta(u_1)) \right]. \end{aligned}$$

Fejér type inequality.

Corollary 4.3.5. Taking $\theta = 1$ in corollary (5.3.4), result leads to Hermite-Hadamard

type inequality for convex function, as defined in theorem (5.1.1).

Corollary 4.3.6. Considering Fejér type inequalities as defined in (5.3.3), by putting $\theta = 1$ the achieved result leads to Hermite-Hadamard type inequality for (η_1, η_2) -convexity as follows

$$\begin{aligned} & \rho\left(\frac{2u_1 + \eta_1(u_2, u_1)}{2}\right) \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1))^-}^{\alpha, v, p}(u_1)\right) \\ & - \frac{1}{4} \int_{u_1}^{u_1 + \eta_1(u_2, u_1)} \left[(u_1 + \eta_1(u_2, u_1) - s)^{\beta} \mathbb{J}_{\beta, \mu, \pi}^{\alpha, v, p}(w(u_1 + \eta_1(u_2, u_1) - s)^{\alpha}) \right. \\ & \left. + (s - u_1)^{\beta} \mathbb{J}_{\beta, \mu, \pi}^{\alpha, v, p}(w(s - u_1)^{\alpha}) \right] \times \eta(\rho(s), \rho(2u_1 + \eta_1(u_2, u_1) - s)) ds \\ & \leq \frac{1}{2} \left[(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p}(\rho))(u_1 + \eta_1(u_2, u_1)) + \mathcal{J}_{\beta, \mu, \pi, w; (u_1 + \eta_1(u_2, u_1))^-}^{\alpha, v, p}(\rho(u_1)) \right] \\ & \leq \left(\frac{2u_1 + \eta_2(\rho(u_2), \rho(u_1))}{2}\right) \left(\mathcal{J}_{\beta, \mu, \pi, w; (u_1)^+}^{\alpha, v, p}(1(u_1 + \eta_1(u_2, u_1)))\right) \end{aligned}$$

Conclusion

The present study has sought to investigate, generalize, and extend the framework of fractional calculus through the exploration of special functions, generalized fractional operators, and integral inequalities associated with convexity. At its core, the work centered upon the introduction and modification of fractional operators built upon Bessel–Maitland functions and their generalizations, with particular emphasis on the Ghayasuddin fractional operators. Furthermore, the research extended classical and generalized inequalities, such as the Hermite–Hadamard (HH), Fejér type, trapezoidal, and midpoint inequalities, to broader function classes including η_1, η_2 -convex functions. Collectively, these contributions reflect an intersection of functional analysis, special functions, and applied mathematics, wherein novel operators and inequalities have been formulated to enhance analytical flexibility and mathematical applicability. The overarching objective was to address the need for refined mathematical tools that transcend the limitations of classical calculus and conventional inequalities. The study recognized that the standard fractional operators, though historically significant, often lack the capacity to fully accommodate generalized special functions and the nuanced structure of convex functions. By introducing new operators that incorporate kernels built upon extended forms of the Bessel–Maitland function, this research has provided a set of instruments that not only generalize existing operators but also unify and extend a variety of inequalities across different mathematical contexts. In this way, the study has consolidated scattered strands of mathematical inquiry into a coherent framework,

producing results with both theoretical depth and practical potential.

One of the key achievements of this work lies in the formulation of the generalized Ghayasuddin fractional operators. These operators, constructed using eight-parameter extensions of the Bessel–Maitland function, embody a remarkable degree of generality. By embedding these highly flexible kernels into the fractional operator framework, the research has been able to recover numerous existing operators as special cases, including the classical Riemann–Liouville, Srivastava, and generalized Srivastava operators. This property of reducibility illustrates the robustness of the newly proposed operators, highlighting their capacity to encapsulate classical forms while simultaneously extending the mathematical landscape with novel structures. Such generalizations are crucial in fractional calculus, where the diversity of applications across physics, engineering, and applied sciences often demands flexible operators that can adapt to different functional contexts. Parallel to the construction of these operators, the study made substantive contributions to the theory of integral inequalities. Inequalities such as Hermite–Hadamard and Fejér type have long been central tools in analysis, optimization, and approximation theory. However, their classical formulations are confined to specific types of convexity. By extending these inequalities to η_1 , η_2 -convex functions and by applying the generalized Ghayasuddin fractional operators, the study has not only broadened the domain of validity for these inequalities but also generated refinements and new variants, including generalized trapezoidal and midpoint forms. These results constitute a significant enrichment of the inequality literature, opening new avenues for mathematical exploration and application.

Another major accomplishment was the rigorous exploration of interrelationships among classical inequalities. By demonstrating how different forms of inequalities—Hermite–Hadamard, Fejér, trapezoidal, and midpoint—are interlinked within the framework of generalized fractional operators, the study highlighted the underlying unity of these analytical constructs. This interconnectedness underscores the deep structural relationships that pervade integral inequalities, suggesting that many results that appear isolated in classical literature can, in fact, be derived as corollaries or special cases of a broader unified framework. Such unification is not merely of theoretical elegance; it offers a methodological advantage, enabling researchers to derive multiple inequalities systematically rather than approaching each case independently. A further noteworthy

dimension of this research has been the comprehensive treatment of Bessel–Maitland functions and their extensions. Starting from the classical Bessel functions and tracing their generalizations through the Bessel–Maitland and extended Wright-type functions, the study presented a systematic account of how these special functions evolve and how their kernel properties can be harnessed in fractional operators. The introduction of six-parameter and higher-order generalizations of Bessel–Maitland functions provided the foundation for constructing new integral operators and for analyzing their properties in depth. This treatment not only expands the catalog of special functions within fractional calculus but also provides a resource for future work in applied mathematics, where such functions frequently appear in the modeling of oscillatory, wave, and diffusion phenomena. From a methodological perspective, the study demonstrated a rigorous balance between theoretical derivation and practical generalization. The process of deriving inequalities was not confined to abstract manipulation but was systematically linked to functional classes, operator properties, and convexity conditions. This methodological robustness ensures that the results are not only mathematically sound but also practically adaptable to different research contexts. By ensuring that the newly developed operators reduce to well-known forms under appropriate parameter restrictions, the study validated the internal consistency of its generalizations, thereby confirming that the new results are not arbitrary extensions but rather natural continuations of established mathematical traditions.

In terms of broader contributions, the study highlights several implications. First, the refinement and generalization of inequalities contribute to optimization theory, particularly in contexts where convexity plays a central role, such as mathematical programming, economics, and control theory. The extension to η_1 , η_2 -convexity allows for greater flexibility in modeling problems where strict convexity may not hold, thereby broadening the scope of optimization tools. Second, the introduction of generalized operators enriches the field of fractional differential equations, as such operators can serve as analytical tools in the modeling of physical processes governed by memory, hereditary effects, or non-local interactions. Third, the systematic exploration of special functions provides resources for computational and applied mathematics, particularly in physics and engineering problems where generalized Bessel-type functions naturally arise. Moreover, the study has provided a foundation for subsequent

mathematical investigations into more generalized forms of convexity and inequalities. By situating the Hermite–Hadamard and Fejér inequalities within the framework of η_1 , η_2 -convex functions, the research implicitly suggests that other inequalities could likewise be extended using alternative generalizations of convexity. This insight creates a fertile ground for future mathematical exploration, where generalized convexities—such as invexity, preinvexity, and higher-order convexities—can be systematically studied through the lens of fractional operators. While the contributions are substantial, it is also important to acknowledge the limitations inherent in the study. One limitation lies in the level of abstraction required for the generalizations. Although the extended operators and inequalities are mathematically elegant and theoretically robust, their practical applications to real-world models remain to be more fully explored. Bridging the gap between abstract generalizations and applied contexts is a challenge that warrants future investigation.

Another limitation is the computational complexity introduced by multi-parameter special functions. While these functions allow for great generality, they also introduce difficulties in explicit computation, which may limit their direct applicability in numerical simulations or applied problem-solving without further algorithmic development. These limitations are not shortcomings but rather natural consequences of mathematical generalization. They point toward directions for further research. For example, the computational challenges associated with multi-parameter Bessel–Maitland functions suggest a need for the development of efficient numerical algorithms and approximation methods. Similarly, the theoretical generalizations invite interdisciplinary collaborations where applied mathematicians, physicists, and engineers can work together to translate these operators and inequalities into models for real-world systems such as viscoelastic materials, diffusion processes, or wave propagation in complex media. The future research directions emerging from this study are manifold. One promising avenue is the further generalization of convexity-related inequalities. Since convexity is central to many branches of mathematics, extending the framework to incorporate new forms of generalized convexities can yield results of wide applicability. Another direction involves the extension of fractional operators to stochastic or probabilistic contexts, where randomness and uncertainty are integral to system behavior. Incorporating fractional operators with kernels defined by generalized special functions into stochastic calculus

could significantly enrich the mathematical modeling of random processes with memory effects. Additionally, the extension of these operators into higher-dimensional and multivariable settings represents another valuable line of inquiry, as many physical and engineering problems inherently involve multiple variables and dimensions.

The study has presented a rigorous, coherent, and original contribution to the field of fractional calculus and inequalities. By introducing generalized Ghayasuddin fractional operators, refining classical inequalities, extending them to new classes of convexity, and systematically analyzing the interconnections among inequalities, the research has enriched both the theoretical foundations and the methodological tools of applied mathematics. It has successfully demonstrated that the interplay between special functions, fractional operators, and convexity is a fertile ground for innovation, yielding results that are not only mathematically significant but also potentially applicable across diverse scientific and engineering domains. The significance of this work lies not merely in the specific results it has produced but in the broader perspective it offers: that mathematics thrives on the dynamic interplay between generalization and specialization, between abstraction and application. The generalized operators and inequalities developed here exemplify this interplay, standing as both an extension of classical forms and an invitation for future exploration. Through this study, a foundation has been laid for continued advancement in fractional calculus, special functions, and inequality theory, ensuring that these fields remain vibrant, interconnected, and responsive to the evolving challenges of mathematics and its applications.

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Online ISSN

3007-3197

Print ISSN

3007-3189

<http://amresearchreview.com/index.php/Journal/about>

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