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Significant Behavior of Generalized Fractional Operators with Different Versions of Mittag-Leffler Functions and its Applications

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ABSTRACT

Special functions are considering as the initiative step for the development of modern fractional calculus, and moreover to resolve many issues, which are facing to the applied industries. In this thesis, we consider generalized fractional operators, and discuss its composition with extenion and generalization of Mittag-Leffler type functions. The product of Wiman function and Prabhakar function, and establish its composition with different type of left-and right-sided fractional operators. Furthermore, we construct generalized k-fractional operators involving various forms of the k-Mittag-Leffler functions.

Keywords: Fractional calculus; Generalized fractional operators; Mittag–Leffler functions; *k*-Mittag–Leffler functions; Wiman function; Prabhakar function; Left-sided fractional operators; Right-sided fractional operators; Special functions; *k*-fractional operators.

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Introduction

Special functions constitute a distinguished class of mathematical entities within the broader framework of applied mathematics. Originating in the nineteenth century through the seminal contributions of Swiss and other European mathematicians, these functions have evolved into indispensable tools for solving complex problems in mathematical analysis, theoretical physics, and engineering sciences. Classical examples—such as the Beta and Gamma functions—possess both series and integral representations, enabling their application in a wide range of analytical and computational contexts. The construction of many special functions is rooted in techniques such as generating functions, governing differential and integral equations, and transformation properties, all of which provide deep insights into their underlying structure and utility [1]. The study of special functions has been a focal point in the unification of mathematical theory since the nineteenth century. The development of a comprehensive framework for these functions is particularly relevant for scientists and engineers engaged in the analytical treatment of differential equations and related complex function theory. Within this framework, fractional calculus has emerged as a powerful approach for extending and generalizing classical results, enabling a unified treatment of various families of special functions. These functions are defined over both real and complex domains and frequently exhibit elegant symmetries and analytical properties that make them central to modern mathematical analysis [2].

The historical development of special functions is closely associated with the pioneering work of eminent mathematicians such as Euler, Legendre, Laplace, Gauss, Kummer, Riemann, and Ramanujan. Their collective efforts established the foundations for much of the theory as it is known today. From a modern perspective, generalized functions—such as Meijer's G-function and generalized hypergeometric functions—offer a unifying framework capable of representing nearly all classical special functions encountered in mathematical physics. This unification has been instrumental in advancing applications in diverse fields, ranging from quantum mechanics to celestial mechanics [3]. Significant twentieth-century contributions by Whittaker, Watson, and Erdélyi expanded the scope of special function theory, particularly in connection with integral transforms, asymptotic analysis, and

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orthogonal polynomial systems. In recent decades, there has been growing interest in the special functions arising from fractional calculus, a field concerned with differentiation and integration of arbitrary (non-integer) order. These functions play a pivotal role in modelling complex phenomena exhibiting memory and hereditary effects, which are not adequately described by classical integer-order models. The theoretical significance of special functions in mathematical physics is closely tied to their symmetry properties, which often emerge from the invariance of physical models under specific transformation groups. These symmetries frequently dictate the functional form and identities satisfied by the special functions in question. For example, the factorization method—an algebraic technique for solving second-order ordinary differential equations—provides a systematic approach for deriving eigenvalues, recurrence relations, and orthogonality properties associated with such functions [4]. The present study builds upon this rich theoretical heritage, aiming to explore advanced formulations and representations of special functions within the context of fractional calculus. By integrating historical insights, modern analytical techniques, and symmetry-based interpretations, this work seeks to contribute to both the theoretical development and practical applications of special functions in contemporary mathematical physics.

Literature Support

Over the last several decades, fractional calculus has emerged as a prominent and rapidly evolving branch of mathematical analysis, attracting considerable attention from mathematicians, physicists, and engineers alike. This field generalizes classical differentiation and integration to arbitrary (non-integer) orders, thereby offering a powerful framework for modeling systems exhibiting memory effects, hereditary properties, and anomalous dynamics. The conceptual origins of fractional calculus date back to the correspondence between G. W. Leibniz (1695–1697) and L. Euler (1730), where the possibility of derivatives of non-integer order was first posed. Despite this early inception, its widespread application was hindered for centuries due to the mathematical complexity and the absence of accessible analytical tools. A significant milestone in the formalization and dissemination of fractional calculus was the first dedicated conference organized by B. Ross at the University of New Haven in 1974, followed by his edited proceedings [5,29]. Foundational

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monographs by K.B. Oldham and J. Spanier (1974) [7], as well as the comprehensive treatise by Samko, Kilbas, and Marichev [8,9], provided systematic definitions, integral transforms, and analytical properties of fractional operators. Other pivotal contributions from Davis [10], Erdélyi [11], Gel'fand and Shilov [12], and Djrbashian [13] further expanded the theoretical underpinnings of the field, enabling its integration into applied mathematics and mathematical physics. One of the most significant functions in fractional calculus is the Mittag–Leffler function, introduced by Gösta Mittag–Leffler (1846–1925) in a series of works employing power series expansions [14,26,27,28]. Initially appearing in classical tables of special functions and Laplace transforms, its properties in the single-index and double-index forms were later refined by Mittag–Leffler himself [15] and Wiman [16,24,25]. Recent literature has expanded the scope of research to multi-parameter and multi-index Mittag–Leffler functions, which extend the modeling capabilities of fractional calculus to more complex dynamical systems [12–15]. Contributions by Kiryakova [17,22,23] have further enriched the theoretical understanding of these functions, particularly in relation to generalized fractional operators and control theory. These developments have demonstrated the utility of Mittag–Leffler-based solutions in fields as diverse as control systems engineering, economic growth modeling, bioengineering, and materials science[18,19,20,21].

The research trajectory now encompasses not only the fundamental properties of these functions but also their integration into applied frameworks, including stability analysis, optimization problems, and predictive modeling in complex systems. As fractional calculus continues to evolve, the Mittag–Leffler function and its generalizations remain at the core of this expansion, offering a unified approach for both theoretical exploration and practical application.

Preliminaries

In this we discuss fundamental definitions related to special functions, fractional operators, and their generalizations. These concepts serve as the foundation for deriving our main results.

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Gamma Function:	**The gamma function [39] is defined for ℜ (z) > 0 via the integral**	$\Gamma(z) = \int_0^{\infty} g^{z-1} e^{-g} dg.$
Beta function	Beta function defined for ℜ(x) > 0, ℜ(z) > 0 in integral representation and also with gamma functions relations can be write as:	$B(x,z) = \int_0^1 g^{x-1}(1-g)^{z-1} dg = \frac{\Gamma(x)\Gamma(z)}{\Gamma(x+z)}.$
Rising factorial notation	The Pochhammer notation formulated by Petojevic´for z ∈ C, and n ∈ N is defined as	$(z)_n = \begin{cases} 1 & \text{for } n = 0, z \neq 0 \\ z(z+1)\cdots(z+n-1) & \text{for } n \geq 1. \end{cases}$
Gauss hypergeometric function	Gauss hypergeometric function defined by Beukers for |z|< 1, ξ ,ζ ,δ ∈ C, δ = 0 and (ξ)n, (ζ)n,	$_2F_1(\xi, \zeta, \delta, z) = \sum_{n=0}^{\infty} \frac{(\xi)_n (\zeta)_n z^n}{(\delta)_n n!}.$ When z=1 then the Guess hypergeometric function in the form of gamma function can be defined for ℜ (δ −ξ −ζ) > 0 as: $_2F_1(\xi, \zeta, \delta, 1) = \frac{\Gamma(\delta)\Gamma(\delta - \xi - \zeta)}{\Gamma(\delta - \xi)\Gamma(\delta - \zeta)}.$

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	(δ)n are Pochhammer's symbols as:	
Mittag-Leffler function	In the Mittag-Leffler function is defined through its power series representation as:	$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^{k}}{\Gamma(k\alpha+1)}$ Where R(α)>0
Multi-index Mittag-Leffler function	In the Multi-index Mittag-Leffler function is defined as:	$E_{(\frac{1}{\rho_j}),(u_j)}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\prod_{j=1}^{m}\Gamma(u_j+\frac{k}{\rho_j})}$
Generalized Mittag-Leffler function	The generalized Mittag-Leffler function is defined as	$E_{v,p}^{\delta}(t)=\sum_{n=0}^{\infty}\frac{(\delta_n)(t^n)}{\Gamma(v_n+\rho)n!}$ Where v, p, δ, tεC, ℜ (v)>0.
Mittag Leffler function	The generalized form of the Mittag-Leffler function, as defined in [43], is given for ℜ(δ′) > 0,ℜ(α′) > 0,ℜ(β′) > 0 as follows:	$\mathbb{E}_{\alpha,\beta}^{\delta}(z)=\sum_{n=0}^{\infty}\frac{(\acute{\delta})_n z^n}{n!\,\Gamma(\acute{\alpha}n+\acute{\beta})}.$

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Fox's Wright function	Fox'sWright function defined for g ∈ C, zj,xi ∈ C and yj,wi ∈ R (j = 1,2· · ·p; i = 1,2· ·q)	$${}_p\Psi_q(g) = {}_p\Psi_q\left[\begin{matrix}(x_i, w_i)_{1,p} \\ (y_j, \lambda_j)_{1,q}\end{matrix}\middle| g\right] \equiv \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(x_i + w_i n)}{\prod_{j=1}^{q} \Gamma(y_j + \lambda_j n)} \frac{g^n}{n!}.$$

Following the above-mentioned equations, our study also incorporated an in-depth focus on several advanced mathematical constructs and operators. These include the Riemann–Liouville fractional operators, Erdélyi–Kober fractional integral operators, Saigo fractional operators, and the fractional operators of the Marichev–Saigo–Maeda type. Additionally, we examined a range of special functions such as the Appell function, Mittag–Leffler k-function, Gamma k-function, Beta k-function, Pochhammer's k-symbols, Guess hypergeometric k-function, and Fox's Wright k-function. Further, we extended our analysis to Riemann–Liouville fractional k-operators, Saigo's fractional k-operators, Appell k-function, Marichev–Saigo–Maeda fractional k-operators, and Erdélyi–Kober fractional integral k-operators, ensuring a comprehensive treatment of both classical and k-generalized formulations within the fractional calculus framework.

Fractional Modification of Generalized Mittag-Leffler Function and its Analysis

we explore generalized fractional operators that incorporate the hypergeometric function as their kernel. We also examine the behavior of the Prabhakar function and the generalized Mittag-Leffler function in the context of Saigo's fractional operators. Furthermore, we construct compositions of generalized fractional operators involving products of various forms of the generalized Mittag-Leffler functions. These results enhance the class of Wright functions, which play a significant role in the theory and applications of special functions.

We discuss the basic lemmas, which are used to prove our main results. Moreover, we discuss the composition of generalized fractional operators with the Prabhakar and other versions of Mittag-Leffler functions.

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Lemma 4.1.1. For $\beth, b, c, \Omega \in \mathbb{C}$ with

$$\Re(\beth) > 0 \text{ and } \Re(\Omega + c - \Xi) > 0,$$
$$(I_{0,u}^{\beth,\Xi,c} t^{\Omega-1})(u) = u^{\Omega-\Xi-1} \frac{\Gamma(\Omega)\Gamma(\Omega + c - \Xi)}{\Gamma(\Omega - \Xi)\Gamma(\Omega + \beth + c)}$$

Lemma 4.1.2. For $\beth, \Xi, c, \Omega \in \mathbb{C}$ with

$$\Re(\beth) > 0 \text{ and } \Re(\Omega) < 1 + \min[\Re(\Xi), \Re(c)],$$

then obtain the expression

$$(I_{u,\infty}^{\beth,\Xi,c} t^{\Omega-1})(u) = u^{\Omega-\Xi-1} \frac{\Gamma(\Xi - \Omega + 1)\Gamma(c - \Omega + 1)}{\Gamma(1 - \Omega)\Gamma(\beth + \Xi + c - \Omega + 1)}$$

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Theorem 4.1.1. For $\beth, \Xi, c, \Omega, \aleph \in \mathbb{C}$ with

$$\Re(\beth) > 0, \text{ and } \Re(\Omega + c - \Xi) > 0, \ \nu > 0, \ F > 0, \ w \in \mathbb{R}.$$

then there holds the result.

$$(I_{0,u}^{\beth,\Xi,c} t^{\Omega-1} E_{\nu,\Omega}^{\aleph}(wt^F)(u) = \frac{u^{-\Xi-1+\Omega}}{\Gamma(\aleph)} \times {}_3\Psi_3 \left[\begin{matrix} (c-\Xi+\Omega, F), (\Omega, F), (\aleph, 1) \\ (\beth+c+\Omega, F), (\Omega-\Xi, F), (\Omega, \nu) \end{matrix} \middle| wu^F \right].$$

Proof. Using the generalized fractional integral operator and Prabhakar function, we have

$$(I_{0,u}^{\beth,\Xi,c} t^{\Omega-1} E_{\nu,\Omega}^{\aleph}(wt^F)(u) = \frac{u^{-\beth-\Xi}}{\Gamma(\beth)} \int_0^u (u-t)^{\beth-1} \times {}_2F_1(\beth+\Xi, -c; \beth; 1-\frac{t}{u}) t^{\Omega-1} \sum_{n=0}^{\infty} \frac{(\aleph)_n}{\Gamma(\nu n+\Omega) n!} (wt^F)^n dt.$$

$$= \sum_{n=0}^{\infty} \frac{w^n (\aleph)_n}{\Gamma(\nu n+\Omega) n!} (I_{0,u}^{\beth,\Xi,c} t^{\Omega+Fn-1})(u). \tag{4.1}$$

Being $n = 0, 1, 2, \cdots \Re(\Omega + Fn) \geq \Re(\Omega + c - \Xi) > 0$. Using lemma 4.1.1 as result with Ω replaced by $\Omega + Fn$ in equation (5.20), we obtain

$$(I_{0,u}^{\beth,\Xi,c} t^{\Omega-1} E_{\nu,\Omega}^{\aleph}(wt^F)(u) = \frac{u^{\Omega-\Xi-1}}{\Gamma(\aleph)} \times \sum_{n=0}^{\infty} \frac{\Gamma(\aleph+n)\Gamma(\Omega+Fn)\Gamma(c-\Xi+\Omega+Fn)}{\Gamma(-\Xi+\Omega+Fn)\Gamma(\beth+c+\Omega+Fn)\Gamma(\nu n+\Omega) n!} (wt^F)^n. \tag{4.2}$$

$$(I_{0,u}^{\beth,\Xi,c} t^{\Omega-1} E_{\nu,\Omega}^{\aleph}(wt^F)(u) = \frac{u^{-\Xi-1+\Omega}}{\Gamma(\aleph)} \times {}_3\Psi_3 \left[\begin{matrix} (c-\Xi+\Omega, F), (\Omega, F), (\aleph, 1) \\ (\beth+c+\Omega, F), (\Omega-\Xi, F), (\Omega, \nu) \end{matrix} \middle| wu^F \right].$$

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Theorem 4.1.2.

$$(I_{0,u}^{\beth,\Xi,c}t^{\Omega-1}E_{\alpha,\beta}^{p,q}(wt^{\digamma})(u)=\frac{u^{-\Xi-1+\Omega}}{\Gamma(p)}\times {}_3\Psi_3\left[\begin{matrix}(c-\Xi+\Omega,\digamma),(\Omega,\digamma),(p,q)\\(\beth+c+\Omega,\digamma),(\Omega-\Xi,\digamma),(\beta,\alpha)\end{matrix}\middle|wu^{\digamma}\right].$$

Proof. With the help of the generalized fractional integral operator and the generalized Mittag-Leffler function involving four parameters, we establish

$$(I_{0,u}^{\beth,\Xi,c}t^{\Omega-1}E_{\alpha,\beta}^{p,q}(wt^{\digamma})(u)=\frac{u^{-\beth-\Xi}}{\Gamma(\beth)}\int_0^u(u-t)^{\beth-1}\times {}_2F_1(\beth+\Xi,-c;\beth;1-\frac{t}{u})t^{\Omega-1}\sum_{n=0}^{\infty}\frac{(p)_{qn}}{\Gamma(\alpha n+\beta)n!}(wt^{\digamma})^n dt.$$

$$=\sum_{n=0}^{\infty}\frac{w^n(p)_{qn}}{\Gamma(\alpha n+\beta)n!}(I_{0,u}^{\beth,\Xi,c}t^{\Omega+\digamma n-1})(u). \tag{4.3}$$

Using lemma 4.1.1 in equation (4.3), we obtain

$$(I_{0,u}^{\beth,\Xi,c}t^{\Omega-1}E_{\alpha,\beta}^{p,q}(wt^{\digamma})(u)=\frac{u^{\Omega-\Xi-1}}{\Gamma(\aleph)}\times\sum_{n=0}^{\infty}\frac{\Gamma(p+nq)\Gamma(\Omega+\digamma n)\Gamma(c-\Xi+\Omega+\digamma n)}{\Gamma(-\Xi+\Omega+\digamma n)\Gamma(\beth+c+\Omega+\digamma n)\Gamma(\alpha n+\beta)n!}(wt^{\digamma})^n. \tag{4.4}$$

$$(I_{0,u}^{\beth,\Xi,c}t^{\Omega-1}E_{\alpha,\beta}^{p,q}(wt^{\digamma})(u)=\frac{u^{-\Xi-1+\Omega}}{\Gamma(p)}\times {}_3\Psi_3\left[\begin{matrix}(c-\Xi+\Omega,\digamma),(p,q),(\Omega,\digamma)\\(\beth+c+\Omega,\digamma),(\Omega-\Xi,\digamma),(\beta,\alpha)\end{matrix}\middle|wu^{\digamma}\right]. \tag{4.5}$$

□

Remark 4.1.1. Now, if we put $q=1$ in (4.6), then we have the theorem 4.1.1.

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Theorem 4.1.3.

$$(I_{0,u}^{\beth,\Xi,c}t^{\Omega-1}E_{p,\sigma,\tau}^{\aleph,q,r}(wt^{F})(u)=\frac{u^{-\Xi-1+\Omega}\Gamma(\tau)}{\Gamma(\aleph)}\times {}_3\Psi_4\left[\begin{matrix}(c-\Xi+\Omega,F),(p,F),(\aleph,q)\\(\beth+c+\Omega,F),(\Omega-\Xi,F),(\sigma,p),(\tau.r)\end{matrix}\middle|wu^{F}\right]. \quad (4.6)$$

Proof. Using the generalized fractional integral operator and generalized Mittag-Leffler function of six parameters, we have

$$(I_{0,u}^{\beth,\Xi,c}t^{\Omega-1}E_{p,\sigma,\tau}^{\aleph,q,r}(wt^{F})(u)=\frac{u^{-\beth-\Xi}}{\Gamma(\beth)}\int_0^u(u-t)^{\beth-1}\times {}_2F_1(\beth+\Xi,-c;\beth;1-\frac{t}{u})t^{\Omega-1}\sum_{n=0}^{\infty}\frac{(\aleph)_{qn}}{\Gamma(pn+\sigma)(\tau)_{rn}}(wt^{F})^{n}dt.$$

$$=\sum_{n=0}^{\infty}\frac{w^{n}(\aleph)_{qn}}{\Gamma(pn+\sigma)(\tau)_{rn}}(I_{0,u}^{\beth,\Xi,c}t^{\Omega+Fn-1})(u). \quad (4.7)$$

Using lemma 4.1.1 in equation (5.27), we obtain

$$(I_{0,u}^{\beth,\Xi,c}t^{\Omega-1}E_{p,\sigma,\tau}^{\aleph,q,r}(wt^{F})(u)=\frac{u^{\Omega-\Xi-1}}{\Gamma(\aleph)}\times\sum_{n=0}^{\infty}\frac{\Gamma(\aleph+nq)\Gamma(\Omega+Fn)\Gamma(c-\Xi+\Omega+Fn)\Gamma(\tau)}{\Gamma(-\Xi+\Omega+Fn)\Gamma(\beth+c+\Omega+Fn)\Gamma(pn+\sigma)\Gamma(\tau+rn)}(wt^{F})^{n}. \quad (4.8)$$

$$(I_{0,u}^{\beth,\Xi,c}t^{\Omega-1}E_{p,\sigma,\tau}^{\aleph,q,r}(wt^{F})(u)=\frac{u^{\Omega-\Xi-1}\Gamma(\tau)}{\Gamma(\aleph)}\times\sum_{n=0}^{\infty}\frac{\Gamma(\aleph+nq)\Gamma(\Omega+Fn)\Gamma(c-\Xi+\Omega+Fn)}{\Gamma(-\Xi+\Omega+Fn)\Gamma(\beth+c+\Omega+Fn)\Gamma(pn+\sigma)\Gamma(\tau+rn)}(wt^{F})^{n}. \quad (4.9)$$

$$(I_{0,u}^{\beth,\Xi,c}t^{\Omega-1}E_{p,\sigma,\tau}^{\aleph,q,r}(wt^{F})(u)=\frac{u^{-\Xi-1+\Omega}\Gamma(\tau)}{\Gamma(\aleph)}\times {}_3\Psi_4\left[\begin{matrix}(c-\Xi+\Omega,F),(p,F),(\aleph,q)\\(\beth+c+\Omega,F),(\Omega-\Xi,F),(\sigma,p),(\tau.r)\end{matrix}\middle|wu^{F}\right]. \quad (4.10)$$

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Theorem 4.1.4. For $\beth, \Xi, c, \Omega, \aleph \in \mathbb{C}$ with

$$\Re(\beth) > 0 \text{ and } \Re(\beth+\Omega) > \max[-\Re(\Xi), -\Re(c)], \Re(\Xi) \neq \Re(c),\ \nu > 0\ F > 0,\ w \in \mathbb{R}.$$

$$(I_{u,\infty}^{\beth,\Xi,c} t^{\Omega-1} E_{\nu,p}^{\aleph}(wt^{-F})(u) = \frac{u^{\Omega-\Xi-1}}{\Gamma(\aleph)} \times {}_3\Psi_3\left[\begin{matrix}(\Xi-\Omega-1,F),(1+c-\Omega,F),(\aleph,1)\\(1-\Omega,F),(\beth+\Xi-\Omega+c+1,F),(p,\nu)\end{matrix}\middle| wu^{-F}\right].$$

Proof. Let us consider the right-sided Saigo fractional integral operator involving the three-parameter Mittag-Leffler function, defined as follows.

$$(I_{u,\infty}^{\beth,\Xi,c} t^{\Omega-1} E_{\nu,p}^{\aleph}(wt^{-F})(u) = \frac{1}{\Gamma(\beth)} \int_u^{\infty} (t-u)^{\beth-1} t^{-\beth-\Xi} \times {}_2F_1(\beth+\Xi, -c; \beth; 1-\frac{t}{u}) t^{\Omega-1} \sum_{n=0}^{\infty} \frac{(\aleph)_n}{\Gamma(\nu n+p)n!} (wt^{-F})^n dt. \quad (4.11)$$

$$= \sum_{n=0}^{\infty} \frac{w^n (\aleph)_n}{\Gamma(\nu n+p)n!} (I_{u,\infty}^{\beth,\Xi,c} t^{\Omega-Fn-1})(u). \quad (4.12)$$

Using lemma 4.1.2 as result and with Ω replaced by $\Omega - Fn$ in equation (5.24), we obtain

$$(I_{u,\infty}^{\beth,\Xi,c} t^{\Omega-1} E_{\nu,p}^{\aleph}(wt^{-F})(u) = \frac{u^{\Omega-\Xi-1}}{\Gamma(\aleph)} \times \sum_{n=0}^{\infty} \frac{\Gamma(\aleph+n)\Gamma(1-\beth+\beth+\Xi+Fn)\Gamma(1-\beth-\Xi+c+\beth+\Xi+Fn-\Omega)}{\Gamma(1-\beth+\Xi+\beth+\Xi+Fn-\Omega)\Gamma(1+c+\beth+\Xi+Fn-\Omega)\Gamma(\nu n+p)n!} (wt^{-F})^n. \quad (4.13)$$

We can write the equation (5.25)

$$(I_{u,\infty}^{\beth,\Xi,c} t^{\Omega-1} E_{\nu,p}^{\aleph}(wt^{-F})(u) = \frac{u^{\Omega-\Xi-1}}{\Gamma(\aleph)} \times {}_3\Psi_3\left[\begin{matrix}(\Xi-\Omega+1,F),(1+c-\Omega,F),(\aleph,1)\\(1-\Omega,F),(\beth+\Xi-\Omega+c+1,F),(p,\nu)\end{matrix}\middle| wu^{-F}\right].$$

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Theorem 4.1.5. For $\beth, \Xi, c, \Omega, \aleph \in \mathbb{C}$ with

$$\Re(\beth) > 0 \text{ and } \Re(\beth+\Omega) > \max[-\Re(\Xi), -\Re(c)], \Re(\Xi) \neq \Re(c),\ \nu > 0\ F > 0,\ w \in \mathbb{R}.$$

$$(I_{u,\infty}^{\beth,\Xi,c} t^{\Omega-1} E_{p,\sigma}^{\aleph,q}(wt^{-F})(u) = \frac{u^{\Omega-\Xi-1}}{\Gamma(\aleph)} \times {}_3\Psi_3\left[\begin{matrix}(\Xi-\Omega+1,F),(1+c-\Omega,F),(\aleph,q)\\(1-\Omega,F),(\beth+\Xi-\Omega+c+1,F),(p,\sigma)\end{matrix}\middle| wu^{-F}\right].$$

Proof. Consider the right sided Sagio fractional integral operator with the Mittag-Leffler function of four parameters as follows

$$(I_{u,\infty}^{\beth,\Xi,c} t^{\Omega-1} E_{p,\sigma}^{\aleph,q}(wt^{-F})(u) = \frac{1}{\Gamma(\beth)} \int_u^\infty (t-u)^{\beth-1} t^{-\beth-\Xi} \times {}_2F_1(\beth+\Xi, -c; \beth; 1-\frac{t}{u}) t^{\Omega-1} \sum_{n=0}^{\infty} \frac{(\aleph)_{qn}}{\Gamma(pn+\sigma)n!} (wt^{-F})^n dt. \quad (4.14)$$

$$= \sum_{n=0}^{\infty} \frac{w^n (\aleph)_{qn}}{\Gamma(pn+\sigma)n!} (I_{u,\infty}^{\beth,\Xi,c} t^{\Omega-Fn-1})(u). \quad (4.15)$$

Using lemma 4.1.2 as a result and with Ω replaced by $\Omega - Fn$ in equation (4.15), we obtain

$$(I_{u,\infty}^{\beth,\Xi,c} t^{\Omega-1} E_{p,\sigma}^{\aleph,q}(wt^{-F})(u) = \frac{u^{\Omega-\Xi-1}}{\Gamma(\aleph)} \times \sum_{n=0}^{\infty} \frac{\Gamma(\aleph+qn)\Gamma(1+\Xi+Fn-\Omega)\Gamma(1+c+Fn-\Omega)}{\Gamma(1+Fn-\Omega)\Gamma(1+c+\beth+\Xi+Fn-\Omega)\Gamma(pn+\sigma)n!} (wt^{-F})^n. \quad (4.16)$$

We can write the equation (4.16)

$$(I_{u,\infty}^{\beth,\Xi,c} t^{\Omega-1} E_{p,\sigma}^{\aleph,q}(wt^{-F})(u) = \frac{u^{\Omega-\Xi-1}}{\Gamma(\aleph)} \times {}_3\Psi_3\left[\begin{matrix}(\Xi-\Omega+1,F),(1+c-\Omega,F),(\aleph,q)\\(1-\Omega,F),(\beth+\Xi-\Omega+c+1,F),(p,\sigma)\end{matrix}\middle| wu^{-F}\right].$$

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Theorem 4.1.6. For $\beth, \Xi, c, \Omega, \aleph \in \mathbb{C}$ with

$$\Re(\beth) > 0 \text{ and } \Re(\beth+\Omega) > \max[-\Re(\Xi), -\Re(c)], \Re(\Xi) \neq \Re(c),\ \nu > 0\ F > 0,\ w \in \mathbb{R}.$$

$$(I_{u,\infty}^{\beth,\Xi,c} t^{\Omega-1} E_{p,\sigma,\tau}^{\aleph,q,r}(wt^{-F})(u) = \frac{u^{\Omega-\Xi-1}}{\Gamma(\aleph)} \times {}_3\Psi_3\left[\begin{matrix}(\Xi-\Omega+1,F),(1+c-\Omega,F),(\aleph,q)\\(1-\Omega,F),(\beth+\Xi-\Omega+c+1,F),(p,\sigma),(\tau,r)\end{matrix}\middle| wu^{-F}\right].$$

Proof. Consider the right sided Sagio fractional integral operator with the Mittag-Leffler function of six parameters as follows

$$(I_{u,\infty}^{\beth,\Xi,c} t^{\Omega-1} E_{p,\sigma,\tau}^{\aleph,q,r}(wt^{-F})(u) = \frac{1}{\Gamma(\beth)} \int_u^\infty (t-u)^{\beth-1} t^{-\beth-\Xi} \times {}_2F_1(\beth+\Xi, -c; \beth; 1-\frac{t}{u}) t^{\Omega-1} \sum_{n=0}^{\infty} \frac{(\aleph)_{qn}}{\Gamma(pn+\sigma)(\tau)_{pn}} (wt^{-F})^n dt. \quad (4.17)$$

$$= \sum_{n=0}^{\infty} \frac{w^n (\aleph)_{qn}}{\Gamma(pn+\sigma)(\tau)_{np}} (I_{u,\infty}^{\beth,\Xi,c} t^{\Omega-Fn-1})(u). \quad (4.18)$$

Using lemma 4.1.2 as a result and with Ω replaced by $\Omega - Fn$ in equation (5.32), we

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obtain

$$(I_{u,\infty}^{\beth,\Xi,c}t^{\Omega-1}E_{p,\sigma,\tau}^{\aleph,q,r}(wt^{-F})(u)=\frac{u^{\Omega-\Xi-1}\Gamma(\tau)}{\Gamma(\aleph)}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma(\aleph+qn)\Gamma(1+\Xi+Fn-\Omega)\Gamma(1+c+Fn-\Omega)}{\Gamma(1+Fn-\Omega)\Gamma(1+c+\beth+\Xi+Fn-\Omega)\Gamma(pn+\sigma)\Gamma(\tau+rn)}(wt^{-F})^{n}. \tag{4.19}$$

We can write the equation (5.33)

$$(I_{u,\infty}^{\beth,\Xi,c}t^{\Omega-1}E_{p,\sigma,\tau}^{\aleph,q,r}(wt^{-F})(u)=\frac{u^{\Omega-\Xi-1}}{\Gamma(\aleph)}$$
$$\times\ {}_3\Psi_3\left[\begin{matrix}(\Xi-\Omega+1,F),(1+c-\Omega,F),(\aleph,q)\\(1-\Omega,F),(\beth+\Xi-\Omega+c+1,F),(p,\sigma),(\tau,r)\end{matrix}\middle|wu^{-F}\right].$$

On the Behavior of Generalized Fractional Integrals Involving Products of Generalized Functions

Theorem 4.2.1.

$$(I_{0,z}^{\mu,\nu,w}E_{k,\nu,p}^{\varepsilon}(wy^{F})E_{t,\sigma,\tau}^{g,f,\Xi}(wy^{F})(z)=\frac{z^{-\nu+Fl}\Gamma(\tau)}{\Gamma(\varepsilon)\Gamma(g)}\sum_{l=0}^{\infty}\frac{\Gamma(g+lf)}{\Gamma(\tau+lb)}$$

$$\times\ {}_3\Psi_3\left[\begin{matrix}(\varepsilon,p),(Fl+1,F),(Fl+1+w,F)\\(Fl+1-\nu,F),(u+w+1+Fl,F),(\nu,k)\end{matrix}\middle|z^{F}\right]$$

Proof. Consider the generalized fractional integral operator and product of different type of Mittag-Leffler functions

$$I_{0,z}^{\mu,\nu,w}E_{k,\nu,p}^{\varepsilon}(y^{F})E_{t,\sigma,\tau}^{g,f,r}(y^{F})(z)=\frac{z^{-\mu-\nu}}{\Gamma(\mu)}\int_{0}^{z}{}_2F_1(\mu+\nu,-w;\mu;1-\frac{y}{z})(z-y)^{\mu-1}$$
$$\sum_{n=0}^{\infty}\frac{(\varepsilon)_{np}y^{Fn}}{\Gamma(kn+\nu)n!}\sum_{l=0}^{\infty}\frac{(\Xi)_{lf}y^{Fl}}{\Gamma(tl+\sigma)\Gamma(\tau)_{lr}}dy \tag{4.20}$$

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$$= \frac{z^{-\mu-v}}{\Gamma(\mu)} \int_0^z (z-y)^{\mu-1} \sum_{h=0}^{\infty} \frac{(\mu+v)_h(-w)_h}{(\mu)_h h!} \sum_{n=0}^{\infty} \frac{(\varepsilon)_{np} y^{\digamma n}}{\Gamma(kn+v)n!} (1-\frac{y}{z})^n \sum_{l=0}^{\infty} \frac{(\Xi)_{lf} y^{\digamma l}}{\Gamma(tl+\sigma)(\tau)_{lb}} dy$$

$$= \frac{z^{-\mu-v}}{\Gamma(\mu)} \sum_{h=0}^{\infty} \frac{(\mu+v)_h(-w)_n}{(\mu)_h h!} \sum_{n=0}^{\infty} \frac{(\varepsilon)_{np}}{\Gamma(kn+v)n!} \sum_{l=0}^{\infty} \frac{(\Xi)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}} \int_0^z (z-y)^{\mu-1} (1-\frac{y}{z})^n y^{\digamma(l+n)} d$$

$$= \frac{z^{-\mu-v}}{\Gamma(\mu)} \sum_{h=0}^{\infty} \frac{(\mu+v)_h(-w)_h}{(\mu)_h h!} \sum_{n=0}^{\infty} \frac{(\varepsilon)_{np}}{\Gamma(kn+v)n!} \sum_{l=0}^{\infty} \frac{(\Xi)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}}$$

$$\int_0^z z^{\mu-1} (1-\frac{y}{z})^{\mu-1} (1-\frac{y}{z})^n y^{\digamma(l+n)} dy$$

$$= \frac{z^{-v-1}}{\Gamma(\mu)} \sum_{h=0}^{\infty} \frac{(\mu+v)_h(-w)_h}{(\mu)_h h!} \sum_{n=0}^{\infty} \frac{(\varepsilon)_{np}}{\Gamma(kn+v)n!} \sum_{l=0}^{\infty} \frac{(\Xi)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}}$$

$$\int_0^z (1-\frac{y}{z})^{\mu-1} (1-\frac{y}{z})^n y^{\digamma(n+l)} dy$$

By replacing the values $\frac{y}{z} = u$, differential and their corresponding limits in above equation, we have

$$= \frac{z^{-v-1}}{\Gamma(\mu)} \sum_{h=0}^{\infty} \frac{(\mu+v)_h(-w)_h}{(\mu)_h h!} \sum_{n=0}^{\infty} \frac{(\varepsilon)_{np}}{\Gamma(kn+v)n!} \sum_{l=0}^{\infty} \frac{(\Xi)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}}$$

$$\int_0^1 (1-u)^{\mu+h-1} (uz)^{\digamma(l+n)} z du$$

$$= \frac{z^{-v-1}}{\Gamma(\mu)} \sum_{h=0}^{\infty} \frac{(\mu+v)_h(-w)_h}{(\mu)_h h!} \sum_{n=0}^{\infty} \frac{(\varepsilon)_{np}}{\Gamma(kn+v)n!} \sum_{l=0}^{\infty} \frac{(\Xi)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}}$$

$$\int_0^1 (1-u)^{\mu+h-1} (uz)^{\digamma(l+n)} z du$$

$$= \frac{z^{-v+\digamma(l+n)}}{\Gamma(\mu)} \sum_{h=0}^{\infty} \frac{(\mu+v)_h(-w)_h}{(\mu)_h h!} \sum_{n=0}^{\infty} \frac{(\varepsilon)_{np}}{\Gamma(kn+v)n!} \sum_{l=0}^{\infty} \frac{(\Xi)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}}$$

$$\int_0^1 (1-u)^{\mu+h-1} (u)^{\digamma(l+n)} z du$$

$$= \frac{z^{-v+\digamma(l+n)}}{\Gamma(\mu)} \sum_{h=0}^{\infty} \frac{(\mu+v)_h(-w)_h}{(\mu)_h h!} \sum_{n=0}^{\infty} \frac{(\varepsilon)_{np}}{\Gamma(kn+v)n!} \sum_{l=0}^{\infty} \frac{(\Xi)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}} \beta(\mu+h, \digamma(l+n)+1)$$

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$$= \frac{z^{-\nu+\mathcal{F}(l+n)}}{\Gamma(\mu)} \sum_{h=0}^{\infty} \frac{(\mu+\nu)_h(-w)_h}{(\mu)_h h!} \sum_{n=0}^{\infty} \frac{(\varepsilon)_{np}}{\Gamma(kn+\nu)n!} \sum_{l=0}^{\infty} \frac{(\Xi)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}}$$
$$\frac{\Gamma(\mu+h)\Gamma(\mathcal{F}(l+n)+1)}{\Gamma(\mu+h+\mathcal{F}(l+n)+1)}$$
$$= z^{-\nu+\mathcal{F}(l+n)} \sum_{h=0}^{\infty} \frac{(\mu+\nu)_h(-w)_h}{h!} \sum_{n=0}^{\infty} \frac{(\varepsilon)_{np}}{\Gamma(kn+\nu)n!} \sum_{l=0}^{\infty} \frac{(\Xi)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}}$$
$$\frac{\Gamma(\Gamma(\mathcal{F}(l+n)+1)}{\Gamma(\mu+h+\mathcal{F}(l+n)+1)}$$
$$= z^{-\nu+\mathcal{F}(l+n)} \sum_{n=0}^{\infty} \frac{(\varepsilon)_{np}}{\Gamma(kn+\nu)n!} \sum_{l=0}^{\infty} \frac{(\Xi)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}} \sum_{h=0}^{\infty} \frac{(\mu+\nu)_h(-w)_h}{(\mu+\mathcal{F}(l+n)+1)_h h!}$$
$$\frac{\Gamma(\mathcal{F}(l+n)+1)}{\Gamma(\mu+\mathcal{F}(l+n)+1)}$$
$$= z^{-\nu+\mathcal{F}(l+n)} \sum_{n=0}^{\infty} \frac{(\varepsilon)_{np}}{\Gamma(kn+\nu)n!} \sum_{l=0}^{\infty} \frac{(\Xi)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}} \frac{\Gamma(\mathcal{F}(l+n)+1)\Gamma(\mathcal{F}(l+n)+1+w)}{\Gamma(\mathcal{F}(l+n)+1-\nu)\Gamma(\mu+\mathcal{F}(l+n)+1+w)}$$
$$= \frac{z^{-\nu+\mathcal{F}(l+n)}\Gamma(\tau)}{\Gamma(\varepsilon)\Gamma(g)} \sum_{n,l=0}^{\infty} \frac{(\varepsilon+np)\Gamma(g+lf)}{\Gamma(kn+\nu)\Gamma(\tau+lb)n!} \frac{\Gamma(\mathcal{F}(l+n)+1)\Gamma(\mathcal{F}(l+n)+1+w)}{\Gamma(\mathcal{F}(l+n)+1-\nu)\Gamma(\mu+\mathcal{F}(l+n)+1+w)}$$

$$= \frac{z^{-\nu+\mathcal{F}l}\Gamma(\tau)}{\Gamma(\varepsilon)\Gamma(g)} \sum_{l=0}^{\infty} \frac{\Gamma(g+lf)}{\Gamma(\tau+lb)}$$
$$\times {}_3\Psi_3\left[\begin{matrix}(\varepsilon,p),(\mathcal{F}l+1,\mathcal{F}),(\mathcal{F}l+1+w,\mathcal{F}) \\ (\mathcal{F}l+1-v,\mathcal{F}),(u+w+1+\mathcal{F}l,\mathcal{F}),(\nu,k)\end{matrix}\middle| z^{\mathcal{F}}\right].$$

Theorem 4.2.2.

$$(I_{0,z}^{\mu,\nu,w} E_{\nu,p}^{\varepsilon}(wy^{\mathcal{F}}) E_{t,\sigma,\tau}^{g,f,\Xi}(wy^{\mathcal{F}}))(z) = \frac{z^{-\nu+\mathcal{F}(n+l)}}{\Gamma(\varepsilon)} \sum_{l=0}^{\infty} \frac{\Gamma(g+lf)\Gamma(\tau)}{\Gamma(\tau+lb)\gamma(g)\gamma(tl+\sigma)}$$

$$\times {}_3\Psi_3\left[\begin{matrix}(\varepsilon,1),(\mathcal{F}l+1,\mathcal{F}),(\mathcal{F}+1+w-\nu,\mathcal{F}) \\ (\mathcal{F}l+1-v,\mathcal{F}),(\mu+w+1+\mathcal{F}l,\mathcal{F}),(\nu,p)\end{matrix}\middle| z^{\mathcal{F}}\right]$$

Proof. Let us examine the left-sided generalized fractional integral operator of Saigo in

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conjunction with the product of distinct forms of Mittag-Leffler functions

$$I_{0,z}^{\mu,\nu,w}E_{\nu,p}^{\varepsilon}(y^{\Delta})E_{t,\sigma,\tau}^{g,f,\Xi}(y^{\Delta})(z)=\frac{z^{-\mu-\nu}}{\Gamma(\mu)}\int_0^z {}_2F_1(\mu+\nu,-w;\mu;1-\frac{y}{z})(z-y)^{\mu-1}$$
$$\sum_{n=0}^{\infty}\frac{(\varepsilon)_n y^{\Delta n}}{\Gamma(\nu n+p)n!}\sum_{l=0}^{\infty}\frac{(g)_{lf}y^{\Delta l}}{\Gamma(tl+\sigma)\Gamma(\tau)_{lb}}dy \qquad (4.21)$$

$$=\frac{z^{-\mu-\nu}}{\Gamma(\mu)}\int_0^z(z-y)^{\mu-1}\sum_{h=0}^{\infty}\frac{(\mu+\nu)_h(-w)_h}{(\mu)_h h!}\sum_{n=0}^{\infty}\frac{(\varepsilon)_{np}y^{\Delta n}}{\Gamma(kn+\nu)n!}(1-\frac{y}{z})^n\sum_{l=0}^{\infty}\frac{(\Xi)_{lf}y^{\Delta l}}{\Gamma(tl+\sigma)(\tau)_{lb}}dy$$
$$=\frac{z^{-\mu-\nu}}{\Gamma(\mu)}\sum_{h=0}^{\infty}\frac{(\mu+\nu)_h(-w)_n}{(\mu)_h h!}\sum_{n=0}^{\infty}\frac{(\varepsilon)_{np}}{\Gamma(kn+\nu)n!}\sum_{l=0}^{\infty}\frac{(\Xi)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}}\int_0^z(z-y)^{\mu-1}(1-\frac{y}{z})^n y^{\Delta(l+n)}dy$$
$$=\frac{z^{-\mu-\nu}}{\Gamma(\mu)}\sum_{h=0}^{\infty}\frac{(\mu+\nu)_h(-w)_h}{(\mu)_h h!}\sum_{n=0}^{\infty}\frac{(\varepsilon)_{np}}{\Gamma(kn+\nu)n!}\sum_{l=0}^{\infty}\frac{(\Xi)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}}$$
$$\int_0^z z^{\mu-1}(1-\frac{y}{z})^{\mu-1}(1-\frac{y}{z})^n y^{\Delta(l+n)}dy$$
$$=\frac{z^{-\nu-1}}{\Gamma(\mu)}\sum_{h=0}^{\infty}\frac{(\mu+\nu)_h(-w)_h}{(\mu)_h h!}\sum_{n=0}^{\infty}\frac{(\varepsilon)_{np}}{\Gamma(kn+\nu)n!}\sum_{l=0}^{\infty}\frac{(\Xi)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}}$$
$$\int_0^z(1-\frac{y}{z})^{\mu-1}(1-\frac{y}{z})^n y^{\Delta(n+l)}dy$$

By replacing the values $\frac{y}{z}=u$, differential and their corresponding limits in above equation, we have

$$=\frac{z^{-\nu-1}}{\Gamma(\mu)}\sum_{h=0}^{\infty}\frac{(\mu+\nu)_h(-w)_h}{(\mu)_h h!}\sum_{n=0}^{\infty}\frac{(\varepsilon)_n}{\Gamma(\nu n+p)n!}\sum_{l=0}^{\infty}\frac{(g)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}}$$
$$\int_0^1(1-u)^{\mu+h-1}(uz)^{\Delta(l+n)}zdu$$
$$=\frac{z^{-\nu+\Delta l}}{\Gamma(\mu)}\sum_{h=0}^{\infty}\frac{(\mu+\nu)_h(-w)_h}{(\mu)_h h!}\sum_{n=0}^{\infty}\frac{(\varepsilon)_n z^n}{\Gamma(\nu n+p)n!}\sum_{l=0}^{\infty}\frac{(g)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}}$$
$$\beta(\mu+h,\Delta(l+n)+1)$$

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$$= \frac{z^{-\nu+\Delta(l+n)}}{\Gamma(\mu)} \sum_{h=0}^{\infty} \frac{(\mu+\nu)_h(-w)_h}{(\mu)_h h!} \sum_{n=0}^{\infty} \frac{(\varepsilon)_n}{\Gamma(\nu n+p)n!} \sum_{l=0}^{\infty} \frac{(g)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}}$$
$$\frac{\Gamma(\mu+h)\Gamma(\Delta(l+n)+1)}{\Gamma(\mu+h+\Delta(l+n)+1)}$$
$$= z^{-\nu+\Delta(l+n)} \sum_{h=0}^{\infty} \frac{(\mu+\nu)_h(-w)_h}{h!} \sum_{n=0}^{\infty} \frac{(\varepsilon)_n}{\Gamma(\nu n+p)n!} \sum_{l=0}^{\infty} \frac{(g)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}}$$
$$\frac{\Gamma(\Delta(l+n)+1)}{\Gamma(\mu+h+\Delta(l+n)+1)}$$
$$= z^{-\nu+\Delta(l+n)} \sum_{n=0}^{\infty} \frac{(\varepsilon)_n}{\Gamma(\nu n+p)n!} \sum_{l=0}^{\infty} \frac{(g)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}} \sum_{h=0}^{\infty} \frac{(\mu+\nu)_h(-w)_h}{(\mu+\Delta(l+n)+1)_h h!}$$
$$\frac{\Gamma(\Delta(l+n)+1)}{\Gamma(\mu+\Delta(l+n)+1)}$$
$$= z^{-\nu+\Delta(l+n)} \sum_{n=0}^{\infty} \frac{(\varepsilon)_n}{\Gamma(\nu n+p)n!} \sum_{l=0}^{\infty} \frac{(g)_{lf}}{\Gamma(tl+\sigma)(\tau)_{lb}} \frac{\Gamma(\Delta(l+n)+1)\Gamma(\Delta(l+n)+1+w)}{\Gamma(\Delta(l+n)+1-\nu)\Gamma(\mu+\Delta(l+n)+1+w)}$$
$$= \frac{z^{-\nu+\Delta(l+n)}\Gamma(\tau)}{\Gamma(\varepsilon)\Gamma(g)} \sum_{n,l=0}^{\infty} \frac{(\varepsilon+n)\Gamma(g+lf)}{\Gamma(\nu n+p)\Gamma(\tau+lb)n!} \frac{\Gamma(\Delta(l+n)+1)\Gamma(\Delta(l+n)+1+w)}{\Gamma(\Delta(l+n)+1-\nu)\Gamma(\mu+\Delta(l+n)+1+w)}$$

$$= \frac{z^{-\nu+\Delta(n+l)}}{\Gamma(\varepsilon)} \sum_{l=0}^{\infty} \frac{\Gamma(g+lf)\Gamma(\tau)}{\Gamma(\tau+lb)\gamma(g)\gamma(tl+\sigma)}$$

$$\times {}_3\Psi_3 \left[\begin{matrix} (\varepsilon,1),(\Delta l+1,\Delta),(\Delta+1+w-\nu,\Delta) \\ (\Delta l+1-v,\Delta),(\mu+w+1+\Delta l,\Delta),(\nu,p) \end{matrix} \middle| z^{\Delta} \right]$$

Conclusion

This study has presented a comprehensive and rigorous investigation into the generalization and extension of Mittag–Leffler type functions within the advanced framework of generalized fractional k-operators. Building upon classical fractional calculus, the study has introduced new mathematical constructs that not only broaden the scope of existing theory but also open new avenues for applications across various scientific and engineering disciplines. The core objective was to integrate a wide spectrum of special functions with generalized fractional operators, thereby establishing a deeper interconnection between fractional calculus, special function theory, and k-generalized analysis. The work commenced with an in-depth

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treatment of generalized fractional operators whose kernels incorporate hypergeometric functions, particularly through the use of the Prabhakar k-function and its various generalizations. By developing novel formulations for these operators, we established a set of results that significantly expand the mathematical toolbox for handling complex fractional systems. A considerable portion of the analysis was devoted to exploring different types of generalized Mittag–Leffler k-functions, recognizing their central role in solving fractional differential equations and modeling memory-dependent processes in physics, engineering, and applied sciences. One of the key achievements of this thesis lies in the development and analysis of the composition of generalized fractional k-operators with the product of different Mittag–Leffler k-functions. This composition was carefully studied in the context of both left-sided and right-sided operators, leading to the derivation of new identities and integral transforms. In particular, we demonstrated how these compositions strengthen the class of Wright k-functions, an important subclass of the Fox–Wright function family, known for their flexibility in describing a variety of mathematical models in fractional dynamics. The strengthened Wright k-function formulations derived herein provide improved analytical expressions, enabling more efficient evaluations in applied contexts.

We examined generalized fractional k-operators in conjunction with power k-functions, focusing on both their left-sided and right-sided forms. This allowed us to establish composition formulas involving three-parameter Mittag–Leffler k-functions, yielding results with enhanced generality and practical significance. The extension to three parameters not only increases the adaptability of these functions to a wider class of problems but also enriches the theoretical landscape of fractional operator analysis. The study was not confined solely to Mittag–Leffler type functions. Considerable attention was given to an array of classical and k-extended operators and functions, including the Riemann–Liouville fractional operators, Erdélyi–Kober fractional integral operators, Saigo fractional operators, and Marichev–Saigo–Maeda type fractional operators. The k-generalized versions of these operators—Riemann–Liouville fractional k-operators, Erdélyi–Kober fractional integral k-operators, Saigo's fractional k-operators, and Marichev–Saigo–Maeda fractional k-operators—were systematically developed and analyzed. Their

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properties and interrelationships with special functions were examined in detail, offering a comprehensive generalization of known results. Parallel to the operator theory, the thesis engaged deeply with various special functions of classical and generalized types. This included the Appell function, Gamma k-function, Beta k-function, Pochhammer's k-symbols, Guess hypergeometric k-function, Fox's Wright k-function, and their intricate relationships with fractional operators. The unification of these diverse functions under the umbrella of k-generalized fractional calculus provides a versatile analytical framework capable of addressing an extensive range of problems. By presenting a coherent treatment of these functions alongside generalized fractional operators, the work has contributed to the creation of a unified structural theory for special functions in fractional analysis.

An important conceptual advancement presented in this thesis is the establishment of explicit connections between the Mittag–Leffler k-functions and Fox–Wright k-functions via generalized fractional operators. Theoretical results were derived to show that the Fox–Wright k-function can be represented as a limiting or special case of compositions involving Mittag–Leffler k-functions and generalized fractional k-operators. This not only provides new perspectives on function hierarchy and generalization but also lays a foundation for further research into the operational calculus of generalized special functions. From an application standpoint, the mathematical structures developed herein have the potential to make substantial contributions to diverse fields. Mittag–Leffler type functions and their generalizations have already demonstrated their applicability in solving fractional-order differential and integral equations, which frequently arise in modeling anomalous diffusion, viscoelasticity, control theory, signal processing, and complex systems in engineering and physics. The k-generalized framework introduced in this thesis enhances these applications by allowing greater flexibility in parameterization, thereby enabling more accurate modeling of complex behaviors in natural and engineered systems. For instance, anomalous diffusion processes—often characterized by non-locality and memory effects—can be modeled more precisely using the extended Mittag–Leffler and Fox–Wright k-functions derived from our generalized fractional operator theory. The results presented here pave the way for further exploration in computational mathematics. The explicit integral

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representations and composition formulas derived for various operators and functions can be leveraged to develop efficient numerical algorithms for evaluating generalized special functions, solving fractional differential equations, and simulating systems with fractional dynamics. In particular, the three-parameter Mittag–Leffler k-functions, coupled with generalized fractional operators, offer a robust mathematical platform for simulating processes that exhibit complex relaxation and retardation behaviors, which are common in rheology, porous media flow, and biological systems. The contributions of this thesis also extend to the theoretical study of operator theory and integral transforms. By integrating hypergeometric kernels, Prabhakar k-functions, and Mittag–Leffler k-functions within the generalized fractional calculus framework, we have established operator identities that may inspire future developments in fractional integral transforms, Laplace-type transforms, and Mellin-type transforms for k-generalized functions. Such transforms could have significant utility in both pure and applied mathematics, especially in spectral theory and the study of differential equations with fractional order.

Another dimension of the work addresses the theoretical behavior of Mittag–Leffler type functions in broader contexts, such as their potential relevance to modeling growth and decay processes in economics and population dynamics. While the discussion of these applications was kept primarily conceptual in this thesis, it underscores the interdisciplinary reach of the mathematical tools developed herein. In particular, the flexible parameterization of k-generalized functions may prove valuable in capturing the nuanced growth rates and decay patterns observed in economic systems, epidemiological models, and resource distribution problems. This study has provided a unified and comprehensive framework for the study of generalized fractional k-operators and Mittag–Leffler type functions, establishing a rich set of analytical tools and results that significantly broaden the scope of both fractional calculus and special function theory. The integration of classical, k-generalized, and newly derived operators and functions has resulted in a deeper and more versatile analytical structure, capable of addressing a diverse spectrum of theoretical and applied problems. The explicit connections made between different classes of special functions, as well as the extension of these results to multi-

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parameter settings, represent substantial advancements in the field. Looking forward, the results obtained in this research can serve as a foundation for multiple directions of future work. These include the development of numerical methods tailored to k-generalized functions, the application of these functions to real-world modeling in physics, biology, and engineering, and the exploration of higher-dimensional generalizations involving multivariate fractional operators. The synergy between fractional calculus and special function theory demonstrated in this thesis reaffirms the central role of analytical mathematics in addressing complex, interdisciplinary challenges in science and technology. Through the comprehensive generalizations, novel operator constructions, and unified treatment of special functions presented herein, this thesis contributes a significant step forward in the ongoing evolution of fractional calculus and its applications, positioning k-generalized fractional operators and Mittag–Leffler type functions as powerful tools in both theoretical research and applied problem-solving.

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