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Initial Taylor and Logarithmic Coefficient Bounds for Analytic Functions Associated with Gregory Coefficients

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ABSTRACT

In this paper, we investigate the initial Taylor and logarithmic coefficient bounds for the class of analytic functions defined by the subordination condition $\frac{f(z)}{z} \prec \frac{z}{\log(1+z)}$. By employing the Carathéodory function representation and suitable coefficient inequalities, we derive estimates for the initial Taylor coefficients and the first three logarithmic coefficients.The obtained results provide coefficient estimates for this newly considered function class and illustrate the application of Carathéodory methods in geometric function theory.

Keywords: Analytic Functions, Subordination, Carathéodory Functions, Taylor Coefficients, Logarithmic Coefficients, Coefficient Bounds

Introduction:

The theory of analytic and univalent functions constitutes a fundamental branch of complex analysis and geometric function theory. One of the central problems in this area is the investigation of Taylor coefficients of normalized analytic functions and the establishment of sharp bounds for various coefficient functionals. These problems have received considerable attention due to their importance in understanding the geometric properties of analytic mappings, particularly their growth, distortion, and univalence behavior. In addition, coefficient estimates play an essential role in studying Hankel determinants, inverse coefficients, and logarithmic coefficients, which provide further insight into the structure and geometric characteristics of analytic functions.

Let $U = \{z: |z| < 1\}$ denote the open unit disk, and let A represent the family of normalized analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_{\mathrm{K}}\, z^{\mathrm{K}} \qquad (1.1)$$

satisfying the normalization conditions $f(0) = f'(0) - 1 = 0$. The coefficients of such functions encode important geometric information about their image domains and mapping properties. Consequently, coefficient estimates and coefficient functional have remained fundamental themes in the theory of univalent functions

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[1, 2].

The positivity of the real part of the derivative implies that f is close-to-convex and, in particular, univalent in U, by the classical univalence criterion of Noshiro and Warschawski; see, for example, Duren [1]. This condition also provides a direct analytic link between the function f and the well-studied family of analytic functions having positive real part. The Carathéodory class $\mathcal{P}$ consists of functions analytic in U satisfying $Re\ p(z) > 0$ and $p(0) = 1$. Each $p \in \mathcal{P}$ has an expansion

$$p(z) = 1 + c^1 z + c^2 z^2 + \cdots. \qquad (1.3)$$

For $f \in \mathcal{R}$, the defining condition allows us to write $f'(z) = p(z)$ for some $p \in \mathcal{P}$. Gregory coefficients are decreasing rational numbers 1/2, 1/12, 1/24, 19/720, … . which are similar in their role to the Bernoulli numbers and appear in a large number of problems, especially in those related to the numerical analysis and to the number theory. They first appeared in the works of the Scottish mathematician James Gregory in 1671 [3] and were subsequently rediscovered many times. Among the famous mathematicians who rediscovered them, we find Laplace, Mascheroni, Fontana, Bessel, Clausen, Hermit, Pearson and Fisher. Furthermore, Gregory's coefficients can rightly be considered one of the most frequently rediscovered in mathematics (the last rediscovery dates back to our century), the reason for which in literature we can find them under various names (e.g. reciprocal logarithmic numbers, Bernoulli numbers of the second kind, Cauchy numbers, etc.). For the same reason, their authorship is often attributed to various mathematicians.

In this paper, we considered the generating function of the Gregory coefficients Gn as follows: $\frac{z}{\ln(1+z)} = \Psi(z) = \sum_{n=1}^{\infty} G_n z$. (2) . As it provides various features of functions, determining the upper bound for coefficients has been one of the main areas of study in geometric function theory

The class and its subordination representation:

Let f defined in (1.1) belongs to the class

$$C_G = \left\{ f \epsilon A : \frac{f(z)}{z} \prec \frac{z}{\log(1+z)} \right\} \qquad (2.1)$$

Define

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + \sum_{n=1}^{\infty} p_n z^n \ . \qquad (2.2)$$

Solving for $w(z)$, we obtain

$$w(z) = \frac{p(z) - 1}{p(z) + 1} \qquad (2.3)$$

For analytic functions f and g in U, we say that f is subordinate to g, written as $f \prec g$, if there exists a Schwarz function

$$w(z) = \sum_{k=1}^{\infty} c_{\text{К}}\, z^{\text{К}} \quad , |w(z)| < 1, \quad w(0) = 0. \qquad (2.4)$$

such that $f(z) = g(w(z))$. When g is univalent, this condition is equivalent to $f(0) = g(0)$ and $f(U) \subseteq g(U)$. So by subordination function there exist a Schwarz

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function such that

$$\frac{f(z)}{z} = \frac{w(z)}{\log(1+w(z))} \qquad (2.5)$$

The generating function of Gregory coefficients is

$$\frac{z}{\log(1+z)} = 1 + \frac{1}{2}z - \frac{1}{12}z^2 + \frac{1}{24}z^3 - \frac{19}{720}z^4 + \frac{3}{160}z^5 + \cdots \qquad (2.6)$$

Substituting $Z = w(z)$in (2.6), equation (2.5) becomes

$$\frac{f(z)}{z} = 1 + \frac{1}{2}w(z) - \frac{1}{12}w(z)^2 + \frac{1}{24}w(z)^3 - \frac{19}{720}w(z)^4 + \frac{3}{160}w(z)^5 + \cdots. \qquad (2.7)$$

Now comparing coefficients, we have

$$c_1 = \frac{p_1}{2}$$

$$c_2 = \frac{p_2}{2} - \frac{p_1^2}{4} \qquad (2.8)$$

$$c_3 = \frac{p_3}{2} - \frac{p_1 p_2}{2} + \frac{p_1^3}{8},$$

$$c_4 = \frac{p_4}{2} - \frac{p_1 p_3}{2} - \frac{p_2^2}{4} + \frac{3p_1^2 p_2}{8} - \frac{p_1^4}{16}$$

The following lemmas is needed for the main results.

Lemma 2.1: [6] *If*

$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}\ (p_1 \geq 0)$,

then

$$2p_2 = {p_1}^2 + x(4 - {p_1}^2)$$

$$4p_3 = {p_1}^3 + 2(4 - {p_1}^2)p_1 x - p_1(4 - {p_1}^2)x^2 + 2(4 - {p_1}^2)(1 - |x|^2)y$$

for some $x, y \in \mathbb{C}$ *with* $|x| \leq 1$ *and* $|y| \leq 1$.

Lemma 2.2 [7] *If* $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}\ (p_1 \geq 0)$, *then*

$$|p_n| \leq 2 \quad (n \geq 1)$$

and if $Q \in [0,1]$ *and* $Q(2Q - 1) \leq R \leq Q$, *then*

$|p_3 - 2Qp_1 p_2 + R{p_1}^3| \leq 2$.

Also

$|p_{n+k}$

$\mu p_n p_k| \leq 2\ max\{1, |2\mu - 1|\}$

$$= 2\begin{cases} 1, & for\ 0 \leq \mu \leq 1 \\ |2\mu - 1|, & otherwise \end{cases}$$

The inequalities are taken from [8] [6] and [8] respectively.

Main Results:

Theorem3.1: Let f belongs to Class C_G as defined in (2.1) and of the form (1.1) then

$$|a_2| \leq \frac{1}{2}, |a_3| \leq \frac{1}{2}, |a_4| \leq \frac{1}{2}.$$

All estimates are sharp.

Proof: Since f belongs to Class C_G so by the definition of subordination, there exists a Schwarz function such that

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$$\frac{f(z)}{z} = \frac{w(z)}{\log(1+w(z))}$$

Since

$w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ (3.1)

By using (3.1) in (2.7) and comparing coefficients with

$$\frac{f(z)}{z} = 1 + a_2 z + a_3 z^2 + a_4 z^4 + \cdots,$$

We obtain

$$a_2 = \frac{c_1}{2}$$
$$a_3 = \frac{c_2}{2} - \frac{c_1^2}{12} \quad (3.2)$$
$$a_4 = \frac{c_3}{2} - \frac{c_1 c_2}{6} + \frac{c_1^3}{24},$$
$$a_5 = \frac{c_4}{2} - \frac{2c_1 c_3 + c_2^2}{12} + \frac{c_1^2 c_2}{8} - \frac{19c_1^4}{720}$$

, Using (2.8) in (3.2) we obtain the following expressions in terms of the Carathéodory coefficients:

$$a_2 = -\frac{p_1}{4}$$
$$a_3 = \frac{1}{4}\left(p_2 - \frac{7}{12}p_1^2\right) \quad (3.3)$$
$$a_4 = \frac{1}{4}\left(p_3 - \frac{7}{6}p_1 p_2 + \frac{17}{48}p_1^3\right)$$

Now applying lemma 2 we have the required results. Here for bound a_3 we choose $\mu = \frac{7}{12}$ and for bound a_4 we use

$$|p_3 - 2Qp_1 p_2 + R{p_1}^3| \leq 2$$

And choose $Q = \frac{7}{12}, R = \frac{17}{48}$.

Hence we have the required result.

Theorem 3.2: : Let f belongs to Class C_G as defined in (2.1) and of the form (1.1). if

$$\frac{1}{2} log \frac{f(z)}{z} = \sum_{k=1}^{\infty} \Upsilon_{\mathrm{K}}\, z^{\mathrm{K}} \quad (3.4)$$

Then

$|\Upsilon_1| \leq \frac{1}{4}, \quad |\Upsilon_2| \leq \frac{1}{4}, |\Upsilon_3| \leq \frac{1}{4},$

Proof: Since f belongs to Class C_G so by the definition of subordination, there exists a Schwarz function such that

$$\frac{f(z)}{z} = \frac{w(z)}{\log(1+w(z))}$$

Taking logarithm, we have

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$$\frac{1}{2}log\frac{f(z)}{z}=\frac{1}{2}log\frac{w(z)}{\log(1+w(z))}$$

Using the taylor expansion,

$$log\frac{w(z)}{\log(1+w(z))}=\frac{w}{2}-\frac{5w^2}{24}+\frac{w^3}{8}+\cdots,$$

We obtain

$$\sum_{k=1}^{\infty}\Upsilon_{\kappa}\,z^{\kappa}\quad=\frac{w(z)}{4}-\frac{5w(z)^2}{48}+\frac{w(z)^3}{16}+\cdots, \qquad (3.5)$$

Expanding (2.3) and comparing the coefficients wee have

$$\begin{aligned}\Upsilon_1&=\frac{1}{8}c_1\\ \Upsilon_2&=\frac{1}{8}c_2-\frac{17}{192}c_1^2 \qquad (3.6)\\ \Upsilon_3&=\frac{1}{8}c_3-\frac{17}{96}c_1c_2+\frac{25}{384}c_1^3\end{aligned}$$

Now taking modulus and applying lemma 2 we have the required results. All these bounfds are sharp for the functions

$$p_1(z)=\frac{1+z}{1-z},\qquad p_2(z)=\frac{1+z^2}{1-z^2},\qquad p_3(z)=\frac{1+z^3}{1-z^3}.$$

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