

An Implementation Of Laplace Decomposition Method For Constructing Exact Solutions Of Nonlinear Partial Differential Equations

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ABSTRACT

The majority of real-world issues are formulated using nonlinear PDEs, which are difficult to solve. Thus, it is required to improve mathematical approaches for solving nonlinear PDEs. The Laplace Transform Method is an effective methodology for solving linear PDEs; however, it is not applicable to nonlinear PDEs. The Adomain decomposition method is used for solving both linear and nonlinear PDEs. In this paper exact solutions of nonlinear PDEs are obtained by combining Laplace transform method together with Adomain decomposition method known as Laplace decomposition method (LDM). This is an excellent mathematical tool which can be used directly to all type of ordinary, partial, fractional and integrodifferential equations without unnecessary assumptions, restrictions, linearization or perturbations which may change the physical behavior of the problem. For nonlinear terms Adomain polynomial are obtained. It is observed that in most of cases the series solution converges to exact functions otherwise its few terms provide high degree of

accuracy as compared to numerical solutions.

Keywords: Laplace Decomposition Method, Exact Solution, Nonlinear PDEs.

INTRODUCTION

PDEs are key instruments in understanding the physical systems., offering a mathematical language to depict the evolution of diverse properties across both time and space. These equations emerge organically from the necessity to represent phenomena in disciplines like physics, engineering, and environmental science.

The connection between PDEs and real-world dynamics is deeply rooted in the application of mathematical models based on fundamental physical principles [1-2]. Solving PDEs allows researchers and scientists to gain valuable insights into the behavior of systems, which is critical in fields like climate modeling, aerospace engineering, and quantum mechanics [3]. Numerous phenomena in physics and engineering are characterized by PDEs. These equations are employed to model various phenomena, including wave propagation, heat flow, ecological dynamics, fluid mechanics, plasma physics, quantum mechanics and population growth. Additionally, PDEs are crucial for describing the scattering behavior of chemically reactive substances.

In literature there are many methods to solve linear PDE, such as method of Integral Transform Methods, separation of variables, D-Alembert principle and so on. PDEs offer a numerous method for both analytical and numerical solutions. In academic literature, many authors interested to obtain exact solutions, which are highly valued for their clarity and insight into the underlying physical processes.

There are many methods that can be employed to obtain exact solution of PDEs. Some of notable methods, Sumudu Transform Method Homotopy Analysis Method [8], Variational Iteration Method [9], Daftardar-Jafari Method [5], Characteristic Method, Laplace Iterational Method [11-13], Homotopy Perturbation Method [14], $e^{(-\psi(\zeta))}$ expansion approach [15], D'Alembert's Principle [16] and Integral Transform Method [17] Adomain Decomposition Method [18-19], modified Kudryashov method [20] and so on.

Numerical methods are essential for solving PDEs when exact analytical solutions are not available. Some of notable methods, Domain Decomposition Method [21], Finite Difference Method [18], Finite Element Method [19], Keller Box Method [23], Time Stepping Method [20], Rung-Kutta Method [22] and many more.

The Laplace Transform Method LTM has been used in applied mathematicians and robotics. Laplace transform also gave foundations to many fields like electrical engineering and control theory.

The Laplace Transform is frequently used to transform PDEs into ODEs and then into algebraic equations. Once a solution is obtained, the process typically involves performing the Laplace inversion to obtain the original solution.

The LTM is highly effective tool to solve linear PDEs, but it has limitations when applied to solve nonlinear partial differential equation.

The LTM is most important Integral Transform Techniques but it comes at second position after Fourier Transform, in terms of its usage [24-26].

In the Decomposition Method for linear or nonlinear PDEs, the principle of equating partial solutions is central. This method involves breaking down the differential equation into simpler components, represented by these partial solutions, and then integrating or combining them to form the complete solution [30]. It has been used for linear and nonlinear PDEs [31-32]. This method has high convergence rate. [33].

In this paper our aim is to investigate nonlinear PDEs by combining Laplace Transform Method with Adomain Decomposition Method and that is known as Laplace Decomposition Method (LDM).

This method does not need the linearization, perturbation and unjustified assumption [31-32]. This method can be used directly to linear and nonlinear PDE.

LDM is easy to use and reduces the computational work, provides the exact solutions (If exist) and it converges rapidly.

METHODOLOGY

Laplace Transform method

The Laplace Transform of $\varphi(t)$ is an excellent tool for solving differential equations and is defined as under follows:

$$\mathcal{L}[\varphi(t)] = \int_0^{\infty} \varphi(t) e^{-st} dt = \Phi(s), \quad (1)$$

For all values of s for which the improper integral converges, where $\varphi(t)$ is continuous and of exponential order [34].

Laplace Transform of the partial derivatives are defined as follows:

$$\begin{aligned} \mathcal{L}\left[\frac{\partial w(x, t)}{\partial t}\right] &= s\mathcal{L}[w] - w(x, 0) \\ \mathcal{L}\left[\frac{\partial^2 w(x, t)}{\partial t^2}\right] &= [s^2\mathcal{L}[w]] - sw(x, 0) - w_t(x, 0) \\ \mathcal{L}\left[\frac{\partial w(x, t)}{\partial x}\right] &= \frac{d\bar{w}}{dx} \\ \mathcal{L}\left[\frac{\partial^2 w(x, t)}{\partial x^2}\right] &= \frac{d^2\bar{w}}{dx^2} \end{aligned}$$

Adomain Decomposition Method

Consider a general differenatil equation in operators form as follows

$$\mathcal{L}w + \aleph w + \Re w = \mathcal{G} \quad (2)$$

$\mathcal{L}$ is an invertible linear operstor, $\aleph$ is a nonlinear operator, $\Re$ is the remaining linear operator, $\mathcal{G}$ is a bounded defined function, and u is an unknown function. We construct the inverse operator $\mathcal{L}^{-1}$

$$\mathcal{L}^{-1}\mathcal{L}w + \mathcal{L}^{-1}\aleph w + \mathcal{L}^{-1}\Re w = \mathcal{L}^{-1}\mathcal{G} \quad (3)$$

Selecting $\mathcal{L}$ as an n th-order derivative operator, $\mathcal{L}^{-1}$ transforms an n -fold integration operator. In the ADM solution decomposed as $w = \sum_{n=0}^{\infty} w_n$. When $w_0 = \mathcal{L}^{-1}\mathcal{G} + a$, Equation (3) gives:

$$w = w_0 - \mathcal{L}^{-1}(\aleph w) - \mathcal{L}^{-1}(\Re w)$$

The ADM breaks down the nonlinear term $\aleph w$ into an infinite summation from zero to infinity of Adomain Polynomials as,

$$\mathfrak{N}w = \sum_{n=0}^{\infty} A_n \quad (4)$$

The Adomain polynomial A_n can be obtained by,

$$A_n(w_0, w_1, \dots, w_n) = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} N(\sum_{i=0}^{\infty} \lambda^i w_i) \right]_{\lambda=0}, n = 0, 1, 2, \dots \quad (5)$$

Some Adomain polynomial are defined as follows,

$$\begin{aligned} A_0 &= G(w_0) \\ A_1 &= w_1 G'(w_0) \\ A_2 &= w_2 G'(w_0) + \frac{1}{2!} w_1^2 G''(w_0) \\ A_3 &= w_3 G'(w_0) + w_1 w_2 G''(w_0) + \frac{1}{3!} w_1^3 G'''(w_0) \end{aligned}$$

and so on.

Two significant observations can be made. First A_0 and A_1 on depends on w_0 and w_1 respectively and By recurrence relation,

$$w_{i+1} = -\mathcal{L}^{-1} A_i - \mathcal{L}^{-1} \mathfrak{N}w_i; \quad i \geq 0 \quad (6)$$

By using above recurrence relation all required components can be obtained and most of cases ADM provides exact solution otherwise only few terms of obtained solution will provide the solution with high degree of accuracy as compard to numerical solution. As Laplace transform method cannot be used to solve nonlinear PDEs directly, however if we merge Laplace transform method together with ADM and known as Laplace Decomposition method LDM and it become excellent mathematical tool which can be used to solve nonlinear partial differential equations. In section 2.3 the detial description of the method is explained.

Laplace Decomposition Method

Assume the PDE as follows,

$$\mathcal{M}w(x, t) + \mathfrak{R}w(x, t) + \mathfrak{N}w(x, t) = k(x, t) \quad (7)$$

with initial conditions

$$w(x, 0) = h(x), \quad w_t(x, 0) = \ell(x) \quad (8)$$

In order to solve nonlinear PDE (7-8) following steps are carried out:

Step 1. Apply Laplace Transform Method to both sides of Eq. (7) and by using properties it yields

$$\bar{w}(x, s) = \frac{1}{s} h(x) + \frac{1}{s^2} \ell(x) + \frac{1}{s^2} \mathcal{L}[k(x, t)] - \frac{1}{s^2} \mathcal{L}[\mathfrak{R}w(x, t)] - \frac{1}{s^2} \mathcal{L}[\mathfrak{N}w(x, t)] \quad (9)$$

Step 2. In order to further solve the Eq. (9) we expand the unknown function in the infinte series $w = \sum_{n=0}^{\infty} w_n$ and for the nonlinear term Adomain polynomilas will be derived using Eq.(5) thus it yeilds as follows

$$\begin{aligned} \mathcal{L} \sum_{n=0}^{\infty} w_n(x, t) &= \frac{1}{s} h(x) + \frac{1}{s^2} \ell(x) + \frac{1}{s^2} \mathcal{L}[k(x, t)] - \frac{1}{s^2} \mathcal{L}[\mathfrak{R} \sum_{n=0}^{\infty} u_n] - \\ &\frac{1}{s^2} \mathcal{L}[\mathfrak{N} \sum_{n=0}^{\infty} A_n] \end{aligned} \quad (10)$$

By expanding the summation in Eq. (10) and by comparing the components it yields: (it should be noted that first componet $w_0(x, t)$ will be only those terms which are outside operators)

$$\mathcal{L}[w_0(x, t)] = \frac{1}{s} h(x) + \frac{1}{s^2} \ell(x) + \frac{1}{s^2} \mathcal{L}[h(x, t)] = K(x, s) \quad (11)$$

$$\mathcal{L}[w_1(x, t)] = -\frac{1}{s^2} \mathcal{L}[\mathfrak{R}w_0(x, t)] - \frac{1}{s^2} \mathcal{L}[A_0] \quad (12)$$

$$\mathcal{L}[w_2(x, t)] = -\frac{1}{s^2} \mathcal{L}[\Re w_1(x, t)] - \frac{1}{s^2} \mathcal{L}[A_1] \quad (13)$$

Thus the general recurrence relation will be as follows:

$$\mathcal{L}(w_{n+1}(x, t)) = -\frac{1}{s^2} \mathcal{L}[\Re w_n(x, t)] - \frac{1}{s^2} \mathcal{L}[A_n], n \geq 0, \quad (14)$$

Step 3. Apply Inverse Laplace Transform Method

$$w_0 = K(x, t)$$

$$w_{n+1}(x, t) = -\mathcal{L}^{-1} \left[\frac{1}{s^2} \mathcal{L}[\Re w_n(x, t)] + \frac{1}{s^2} \mathcal{L}(A_n) \right] \quad (15)$$

For solving Eq. (15) for $n \geq 0$ will provide us $w_0, w_1, w_2, \dots$ respectively and then making their sum will provide us closed-form solution of Eq. (7).

APPLICATION OF LAPLACE DECOMPOSITION METHOD

Example 1: Assume the nonlinear PDE

$$w_{tt} = x^2(w_x w_{xx})_x - x^2(w_{xx})^2 - w \quad (16)$$

with initial conditions

$$w(x, 0) = 0, w_t(x, 0) = x^2$$

By employing Laplace transform both sides of Eq. (16) and by using initial condition it yields

$$\mathcal{L}(w(x, t)) = \frac{x^2}{s^2} + \frac{1}{s^2} \mathcal{L}(x^2(w_x w_{xx})_x - x^2(w_{xx})^2 - w) \quad (17)$$

By expanding unknown function in Eq. (17) in infinite series and for nonlinear terms Adomain polynomials are constructed as follows:

$$\mathcal{L}(\sum_{n=0}^{\infty} w_n) = \frac{x^2}{s^2} + \frac{1}{s^2} \mathcal{L}(x^2(\sum_{n=0}^{\infty} A_n)_x - x^2 \sum_{n=0}^{\infty} B_n - \sum_{n=0}^{\infty} w_n) \quad (18)$$

where for nonlinear terms Adomain Polynomials are as follows:

$$\begin{cases} A_0 = w_{0xx} w_{0x} \\ A_1 = w_{0xx} w_{1x} + w_{0x} w_{1xx} \\ A_2 = w_{0xx} w_{2x} + w_{1xx} w_{1x} + w_{2xx} w_{0x} \\ \text{and } B_0 = (w_{0xx})^2 \\ B_1 = 2w_{0xx} w_{1xx} \\ B_2 = 2w_{0xx} w_{2xx} + (w_{1xx})^2 \end{cases}$$

By expanding the Eq. (18) both sides and by collecting the terms outside the operators the first component is as under

$$\mathcal{L}(w_0) = \frac{x^2}{s^2} \quad (19)$$

The general recurrence relations is as follows:

$$\mathcal{L}(w_{k+1}) = \frac{1}{s^2} \mathcal{L}(x^2(A_k)_x - x^2 B_k - w_k) \quad (20)$$

By solving Eq. (20) for $k \geq 0$ and by using Eq. (19) it yields the following components

$$w_0 = x^2 t$$

$$\begin{aligned}w_1 &= -x^2 \frac{t^3}{3!} \\w_2 &= x^2 \frac{t^5}{5!} \\w_3 &= -x^2 \frac{t^7}{7!}\end{aligned}$$

Thus making the sum of all components it yields the following solution In order to get

$$w(x, t) = x^2 t - x^2 \frac{t^3}{3!} + x^2 \frac{t^5}{5!} - \dots$$

(21)

$$w(x, t) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{t^{2k+1}}{(2n+1)!}$$

(22)

It is important to mention that the obtained solution is closed and also satisfies the initial conditions.

Example 2: Assume the nonlinear PDE [6],

$$w_t + \frac{1}{36} x w_{xx}^2 = x^3$$

(23)

with initial conditions

$$w(x, 0) = 0$$

By employing Laplace transform both sides on Eq. (23) and by using initial condition it yields

$$\mathcal{L}(w) = \frac{x^3}{s^2} - \frac{1}{36s} x \mathcal{L}(w_{xx}^2)$$

(24)

By expanding unknown function in Eq. (24) in infinite series and for nonlinear terms Adomain polynomial are constructed as follows:

$$\mathcal{L}(\sum_{n=0}^{\infty} w_n) = \frac{x^3}{s^2} - \frac{1}{36s} x \mathcal{L}(\sum_{n=0}^{\infty} A_n)$$

(25)

where for nonlinear terms Adomain Polynomials are as follows:

$$\begin{cases} A_0 = w_{0xx}^2 \\ A_1 = 2w_{0xx}w_{1xx} \\ A_2 = 2w_{0xx}w_{2xx} + w_{1xx}^2 \end{cases}$$

By expanding the Eq. (25) both sides and by collecting the terms outside the operators the first component is as under:

$$\mathcal{L}(w_0) = \frac{x^3}{s^2}$$

(26)

The general recurrence relations is as follows:

$$\mathcal{L}(w_{k+1}) = -\frac{1}{36s} x \mathcal{L}(A_k)$$

(27)

By solving Eq. (27) for $k \geq 0$ and by using Eq. (28) it yields the following components

$$w_0 = x^3 t$$

$$w_1 = -x^3 \frac{t^3}{3}$$

$$w_2 = x^3 \frac{t^5}{15}$$

Thus making the sum of all components it yields the following solution in order to get

$$w(x, t) = x^3 t - x^3 \frac{t^3}{3} + x^3 \frac{t^5}{15} + \dots$$

(28)

$$w(x, t) = x^3 \sum_{n=0}^{\infty} (-1)^n \frac{t^{2k+1}}{(2n+1)!!}$$

(29)

It is important to mention that the obtained solution is closed and also satisfies the initial conditions.

Example 3: Assume the 2-dimensional nonlinear PDE [33],

$$w_{tt} = (w_{xx}w_{yy})_{xy} - (xyw_xw_y)_{xy} - w$$

(30)

with initial conditions

$$w(x, y, 0) = e^{xy}, w_t(x, y, 0) = e^{xy}$$

By using the same procedure as used in previous examples and we obtained the equations (31)-(32).

where for nonlinear terms Adomain Polynomials are as follows:

$$\begin{cases} A_0 = w_{0xx}w_{0yy} \\ A_1 = w_{0xx}w_{1yy} + w_{0yy}w_{1xx} \\ A_2 = w_{0xx}w_{2yy} + w_{1xx}w_{1yy} + w_{2xx}w_{0yy} \\ \text{and } B_0 = w_{0x}w_{0y} \\ B_1 = w_{0x}w_{1y} + w_{0y}w_{1x} \\ B_2 = w_{0x}w_{2y} + w_{1x}w_{1y} + w_{2x}w_{0y} \end{cases}$$

Similarly using Adomain polynomials and using recurrence relation, we get

$$w_0(x, y, t) = e^{xy}(1 + t)$$

(31)

$$w_{k+1}(x, y, t) = \mathcal{L}^{-1} \left(\frac{1}{s^2} \mathcal{L}[(A_k)_{xy} - (xyB_k)_{xy} - w_k] \right)$$

(32)

Insert $k = 0$ into equation (32), we get

$$w_1(x, y, t) = e^{xy} \left(-\frac{t^2}{2!} - \frac{t^3}{3!} \right)$$

(33)

Insert $k = 1$ into equation (32), we get

$$w_2(x, y, t) = e^{xy} \left(\frac{t^4}{4!} + \frac{t^5}{5!} \right)$$

(34)

Thus making the sum of all components it yields the following solution In order to get

$$w(x, y, t) = e^{xy}(1 + t) + e^{xy} \left(-\frac{t^2}{2!} - \frac{t^3}{3!} \right) + e^{xy} \left(\frac{t^4}{4!} + \frac{t^5}{5!} \right) + \dots$$

(35)

Closed form of above equation can be written as,

$$w(x, y, t) = e^{xy}(\cos(t) + \sin(t))$$

(36)

It is important to mention that the obtained solution is closed and also satisfies the initial conditions.

Example 4: Assume the nonlinear PDE [34],

$$w_t(x, t) + w_x(x, t) + w(x, t)w_x(x, t) - w_{xxt}(x, t) = 0$$

(37)

with initial conditions

$$w(x, 0) = x$$

By using the same procedure as used in previous examples and we obtained the equations (38)-(39).

where for nonlinear terms Adomain Polynomials are as follows:

$$\begin{cases} A_0 = w_{0x}w_0 \\ A_1 = w_{0x}w_1 + w_0w_{1x} \\ A_2 = w_{0x}w_2 + w_{1x}w_1 + w_{2x}w_0 \end{cases}$$

$$\mathcal{L}(w_0(x, t)) = \frac{x}{s}$$

(38)

and

$$\mathcal{L}(w_{k+1}(x, t)) = \frac{1}{s} \mathcal{L}(w_{kxxt}(x, t) - w_{kx}(x, t) - A_k)$$

(39)

By apply Inverse Laplace Transform Method on equations (38) and (39), we get

$$w_0(x, t) = x$$

(40)

And

$$w_{k+1}(x, t) = \mathcal{L}^{-1}\left(\frac{1}{s} \mathcal{L}(w_{kxxt}(x, t) - w_{kx}(x, t) - A_k)\right)$$

(41)

Insert $k = 0$ into equation (41), we get

$$w_1(x, t) = -t - xt$$

(42)

Insert $k = 1$ into equation (41), get

$$w_2(x, t) = t^2 + xt^2$$

(43)

Thus making the sum of all components it yields the following solution In order to get

$$w(x, t) = x - t - xt + t^2 + xt^2 + \dots$$

(44)

By using summation formula of geometric series, we get

$$w(x, t) = \frac{x-t}{1+t}$$

(45)

It is important to mention that the obtained solution is closed and also satisfies the initial conditions.

Example 5: Assume the nonlinear PDE [36],

$$w_t(x, t) = w_{xx}(x, t) + w(x, t) - w^2$$

(46)

Subject to initial conditions

$$w(x, 0) = \mu$$

By using the same procedure as used in previous examples and we obtained the equations (47)-(48).

where for nonlinear terms Adomain Polynomials are as follows:

$$\begin{cases} A_0 = w_0^2 \\ A_1 = w_0 w_1 \\ A_2 = 2w_0 w_2 + w_1^2 \end{cases}$$

$$\mathcal{L}(w_0(x, t)) = \frac{1}{s} \mu$$

(47)

and

$$\mathcal{L}(w_{k+1}(x, t)) = \frac{1}{s} \mathcal{L}(w_{kxx}(x, t) + w_k(x, t) - A_k(x, t))$$

(48)

Use Inverse Laplace Transform Method on equations (47) and (48), we get

$$w_0(x, t) = \mu$$

(49)

And

$$w_{k+1}(x, t) = \mathcal{L}^{-1}\left(\frac{1}{s} \mathcal{L}(w_{kxx}(x, t) + w_k(x, t) - A_k(x, t))\right)$$

(50)

Insert $k = 0$ into equation (50), we get

$$w_1(x, t) = \mu(1 - \mu)t$$

(51)

Insert $k = 1$ into equation (50), we get

$$w_2(x, t) = \mu(1 - \mu)(1 - 2\mu) \frac{t^2}{2!}$$

(52)

Thus, making the sum of all components it yields the following solution in order to get

$$w(x, t) = \mu + \mu(1 - \mu)t + \mu(1 - \mu)(1 - 2\mu) \frac{t^2}{2!} \dots$$

(53)

Using Taylor's series, the exact solution converges to

$$w(x, t) = \frac{\mu e^t}{1 - \mu + \mu e^t}$$

(54)

It is important to mention that the obtained solution is closed and also satisfies the initial conditions.

CONCLUSION

In this article, we explored numerous linear and nonlinear PDEs and developed exact solutions in each instance using the Laplace decomposition method for both homogeneous and inhomogeneous with constant and variable coefficients. The Laplace decomposition technique (LDM) is basic, straightforward, and computationally easy to employ for finding analytical solutions to nonlinear partial differential equations. It is vital to note that LDM does not need any extra

assumptions, restrictions, linearization, or perturbation that might affect the physical state of the original issue and can be applied to any type of problem. In most of the cases the series solutions converge to exact functions otherwise only few terms of obtained solutions provide high degree of accuracy as compare to numerical solutions. This LDM has efficiency to solve fractional differential equations, nonlinear integral equations and integrodifferential equations.

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