

An Intelligent Decision Support Framework for Agricultural Applications Based on Bipolar Spherical Fuzzy Hypersoft Sets

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Abstract

Agriculture plays a vital role in economic development and food security, where effective planning, evaluation, and resource management are essential for achieving sustainable outcomes. Agricultural decision-making often involves complex, uncertain, and sometimes contradictory information arising from multiple criteria and sub-criteria such as soil conditions, climatic factors, crop characteristics, irrigation methods, and environmental constraints. Therefore, reliable decision support tools are required to analyze such information and assist policymakers, researchers, and practitioners in making well-informed choices. To address these challenges, this study proposes a novel Bipolar Spherical Fuzzy Hypersoft Set (BSFHSS) framework, which extends the capability of existing fuzzy models by simultaneously handling positive and negative information under a spherical fuzzy environment. The proposed structure consists of two interconnected spherical fuzzy hypersoft sets, representing favorable and unfavorable information for each degree of membership, non-membership, and hesitancy. Fundamental properties of the proposed model are established, including set-theoretic operations and an in-depth investigation of De Morgans laws within the BSFHSS context. Furthermore, a Bipolar Spherical Fuzzy Hypersoft Weighted Geometric (BSFHSWG) aggregation operator is developed to effectively combine bipolar spherical

fuzzy hypersoft numbers under multiple sub-attributes. Based on the proposed operators, a systematic multi-attribute decision-making algorithm is constructed to support performance evaluation and comparative analysis in agricultural applications. The presented framework demonstrates the significance of incorporating bipolarity, uncertainty, and hierarchical attribute structures when addressing complex agricultural decision-making problems, thereby offering a robust and flexible tool for intelligent agricultural planning and assessment.

Key Words: Spherical Fuzzy Hypersoft Sets, Bipolarity, Agricultural Decision Support Systems, Decision Making, Optimization, Quality Control, Fuzzy Control.

1 Introduction

Making decisions in uncertain environments necessitates mathematical tools capable of handling vagueness and ambiguity. Zadehs fuzzy set theory [1] laid the foundation for modeling uncertainty by allowing partial membership of elements. Since its introduction, fuzzy set theory has been extended in numerous directions and has been successfully applied to a wide range of real-world decision-making problems. One notable extension is intuitionistic fuzzy set theory (IFS), introduced by Atanassov [2], which assigns independent membership and non-membership degrees to an element under the condition that their sum does not exceed one. Although IFS has been widely applied in various domains, it does not explicitly account for neutrality, which is often present in human judgments. In many practical situations, decision makers express opinions that go beyond simple acceptance or rejection. For example, responses such as acceptance, abstention, rejection, and refusal frequently arise in voting systems and survey analysis. Consider a democratic election in which 500 ballots are distributed for a candidate, resulting in 214 votes in favor, 23 abstentions, 123 votes against, and 140 refusals. Abstention reflects neutrality, where voters neither support nor oppose the candidate, while refusal indicates invalid or bypassed voting. Such scenarios highlight the necessity of incorporating neutrality into uncertainty modeling. To address this limitation, Cuong [3] introduced the picture fuzzy set (PFS), which incorporates degrees of acceptance, neutrality, and rejection. PFS effectively represents human opinions involving multiple response categories. However, the constraint $0 \leq \mu + \eta + \nu \leq 1$ limits the independent assignment of these degrees, thereby restricting the expressive power of PFS. To overcome this drawback, Mehmood [4] proposed the spherical fuzzy set (SFS), which generalizes PFS by relaxing the constraint to $0 \leq \mu^2 + \eta^2 + \nu^2 \leq 1$.

This modification significantly enlarges the admissible domain of membership grades, offering greater flexibility in modeling uncertainty. Parallel to these developments, Molodtsov [5] introduced soft set theory as a parameterized mathematical framework for handling uncertainty, providing an alternative to fuzzy and rough set theories. Due to its flexibility, soft set theory has proven effective in decision analysis. Maji [6] defined basic operations on soft sets, which were later enhanced by Ali [7] through extended and restricted unions and intersections, as well as restricted differences. Ali [8] further explored the algebraic structures induced by these operations. Additionally, Aktas [9] incorporated group-theoretic concepts into fuzzy and rough set frameworks, enriching the theoretical foundations of uncertainty modeling. A major advancement was achieved with the introduction of fuzzy soft sets by Maji [10], which combine fuzzy membership values with parameterized soft sets. Deng and Wang [11] proposed the objectparameter approach to manage incomplete information in fuzzy soft environments. Naz [12] investigated algebraic properties of fuzzy soft sets, while Roy [13] demonstrated their effectiveness in decision-making problems. Extending this idea, Maji [14] introduced intuitionistic fuzzy soft sets by integrating intuitionistic fuzzy logic with soft sets, enabling a more comprehensive treatment of uncertainty. To simultaneously model acceptance, neutrality, and rejection within a soft framework, Yolcu [15] proposed picture fuzzy soft sets (PFSS). This model allows alternatives to be evaluated across multiple parameters while explicitly accounting for neutral responses. Similarly, Fathima [16] introduced spherical fuzzy soft sets (SFSS), which provide enhanced flexibility and reliability in decision analysis. Supporting this perspective, Dubois [17] emphasized that effective decision-making requires the simultaneous consideration of both favorable and unfavorable information. In many real-world problems, each positive attribute is naturally associated with a corresponding negative attribute. For instance, the negation of tall is not necessarily short, illustrating the complexity of bipolar human perception. To address such situations, Shabir [18] introduced bipolar soft sets, enabling the simultaneous modeling of positive and negative information. Karaaslan [19] refined this concept by establishing a bijective mapping between parameters and their negative counterparts, while Shabir [20] and Karaaslan [22] explored their algebraic and group-theoretic properties. Ali [21] further extended this framework by proposing spherical fuzzy bipolar soft sets, which integrate bipolarity with spherical fuzzy information. A significant generalization of soft set theory was introduced by Smarandache [23] through hypersoft sets (HS), where the approximation function is defined over multiple sub-attributes rather than a single parameter. This innovation greatly enhances modeling flexibility for complex decision problems. Saeed [24, 25] and Abbas [26] studied various operations on

hypersoft sets, contributing to their theoretical development. Yolcu extended this framework by proposing fuzzy hypersoft sets [27] and intuitionistic fuzzy hypersoft sets [28]. Saeed [29] introduced picture fuzzy hypersoft sets to improve visualization and evaluation in multi-criteria decision-making. Later, Saeed [30] developed picture fuzzy hypersoft graphs to analyze complex systems such as product sale risk. Musa [31] proposed bipolar hypersoft sets by integrating bipolarity with hypersoft structures. Most recently, Harl [32] introduced bipolar picture fuzzy hypersoft sets, combining bipolarity, picture fuzzy logic, and hypersoft parameterization into a unified and highly expressive decision-making framework.

1.1 Research Gap

Current fuzzy and hypersoft models used in agricultural decision-making are not fully capable of handling the complex and uncertain nature of agricultural systems. Most existing approaches consider either uncertainty or multi-attribute information, but they do not adequately represent both positive and negative effects of agricultural factors at the same time. In addition, the influence of sub-attributes such as soil nutrients, climate conditions, and irrigation levels is often treated independently, which reduces the accuracy of real-world analysis. Therefore, there is a clear need for a model that can manage uncertainty, bipolar information, and interrelated sub-attributes within a single framework.

1.2 Main Objective

The main objective of applying the BSFHSS model in agriculture is to provide a reliable mathematical framework for evaluating agricultural problems involving multiple attributes and conflicting information. The model aims to represent favorable, unfavorable, and neutral effects of agricultural factors simultaneously, while also incorporating uncertainty in expert judgments. This approach supports better decision-making in areas such as crop selection, soil quality assessment, irrigation planning, and yield optimization.

1.3 Significant Contribution

The BSFHSS framework contributes to agricultural decision analysis by offering a more flexible and realistic way to model complex agricultural data. It allows decision-makers to distinguish between supportive and harmful factors while considering detailed sub-attributes under uncertain conditions. By combining bipolar representation with spherical fuzzy constraints and

hypersoft parameterization, the proposed approach improves the accuracy and consistency of agricultural evaluations. As a result, BSFHSS serves as an effective tool for enhancing decision-making in modern and sustainable agriculture.

2 Preliminaries

In this section, several fundamental fuzzy algebraic structures are briefly reviewed, as they form the theoretical basis for the development of the spherical fuzzy hypersoft set framework. These concepts facilitate the modeling of uncertainty and multi-attribute information, which are essential for constructing the proposed SFHSS model.

Definition 2.1. ([2]): Let $\mathcal{L}_{\mathbf{k}}$ denote a universe of discourse and $\mathbf{P}(\mathcal{L}_{\mathbf{k}})$ represent its power set. Let $\mathbf{C}$ be a collection of attributes. A pair $(\mathcal{S}, \mathbf{C})$ is called a soft set over $\mathcal{L}_{\mathbf{k}}$ if $\mathcal{S} : \mathbf{C} \rightarrow \mathbf{P}(\mathcal{L}_{\mathbf{k}})$ is a mapping that assigns to each attribute in $\mathbf{C}$ a subset of the universe.

Definition 2.2. ([23]): Let $\mathcal{L}_{\mathbf{k}}$ be a universe of discourse and $\mathbf{P}(\mathcal{L}_{\mathbf{k}})$ its power set. Let $\mathbf{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \dots, \mathbf{c}_n\}$ be a set of n disjoint parameters whose corresponding attribute values are $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \dots, \mathcal{R}_n$ such that $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_3 \times \dots \times \mathcal{R}_n$, with $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ for $i \neq j$, $i, j \in \{1, 2, \dots, n\}$. Then, the pair $(\mathcal{S}, \mathcal{R})$, where $\mathcal{S} : \mathcal{R} \rightarrow \mathbf{P}(\mathcal{L}_{\mathbf{k}})$, is called a hypersoft set over $\mathcal{L}_{\mathbf{k}}$.

Definition 2.3. ([4]): A spherical fuzzy set defined on the universe of discourse $\mathcal{L}_{\mathbf{k}} = \{\mathbf{l}_{\mathbf{k}1}, \mathbf{l}_{\mathbf{k}2}, \dots, \mathbf{l}_{\mathbf{k}n}\}$ is expressed as $\mathcal{V} = \{\langle \mu_{\mathcal{V}}(\mathbf{l}_{\mathbf{k}i}), \eta_{\mathcal{V}}(\mathbf{l}_{\mathbf{k}i}), \nu_{\mathcal{V}}(\mathbf{l}_{\mathbf{k}i}) \rangle \mid \mathbf{l}_{\mathbf{k}i} \in \mathcal{L}_{\mathbf{k}}\}$,

where $\mu_{\mathcal{V}} : \mathcal{L}_{\mathbf{k}} \rightarrow [0, 1]$, $\eta_{\mathcal{V}} : \mathcal{L}_{\mathbf{k}} \rightarrow [0, 1]$, and $\nu_{\mathcal{V}} : \mathcal{L}_{\mathbf{k}} \rightarrow [0, 1]$ represent the degrees of membership, neutral membership, and non-membership, respectively, satisfying $(\mu_{\mathcal{V}}(\mathbf{l}_{\mathbf{k}i}))^2 + (\eta_{\mathcal{V}}(\mathbf{l}_{\mathbf{k}i}))^2 + (\nu_{\mathcal{V}}(\mathbf{l}_{\mathbf{k}i}))^2 \leq 1$.

Definition 2.4. ([18]): Let $\mathcal{L}_{\mathbf{k}}$ be a universal set and $\mathbf{P}(\mathcal{L}_{\mathbf{k}})$ its power set. Let $\mathbf{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \dots, \mathbf{c}_n\}$ be a set of n disjoint parameters, and let $\mathbb{S} \subseteq \mathbf{C}$. The triple $(\mathcal{K}, \mathcal{L}, \mathbb{S})$ is called a *bipolar soft set* over $\mathcal{L}_{\mathbf{k}}$, where $\mathcal{K}$ and $\mathcal{L}$ are functions defined as $\mathcal{K} : \mathbb{S} \rightarrow \mathbf{P}(\mathcal{L}_{\mathbf{k}})$ give favorable information and $\mathcal{L} : \mathbb{S} \rightarrow \mathbf{P}(\mathcal{L}_{\mathbf{k}})$ give unfavorable information such that $\mathcal{K}(s) \cap \mathcal{L}(s) = \emptyset$ for all $s \in \mathbb{S}$.

A bipolar soft set can be represented as $(\mathcal{K}, \mathcal{L}, \mathbb{S}) = \{(s, \mathcal{K}(s), \mathcal{L}(s)) : s \in \mathbb{S} \wedge \mathcal{K}(s) \cap \mathcal{L}(s) = \emptyset\}$.

Definition 2.5. ([31]): Let $\mathcal{L}_{\mathbb{k}}$ be a universal set and let $\mathbf{P}(\mathcal{L}_{\mathbb{k}})$ denote its power set. Let $\mathbf{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \dots, \mathbf{c}_n\}$ be a collection of n mutually disjoint parameters, where the corresponding attribute-value sets are $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \dots, \mathcal{R}_n$ such that $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_3 \times \dots \times \mathcal{R}_n$.

A triple $(\mathcal{K}, \mathcal{L}, A)$ is called a bipolar hypersoft set over $\mathcal{L}_{\mathbb{k}}$ where $\mathcal{K} : \mathcal{R} \rightarrow \mathbf{P}(\mathcal{L}_{\mathbb{k}})$ give favorable information and $\mathcal{L} : \mathcal{R} \rightarrow \mathbf{P}(\mathcal{L}_{\mathbb{k}})$ give unfavorable information, such that $\mathcal{K}(s) \cap \mathcal{L}(s) = \emptyset$ for every $s \in \mathcal{R}$.

A bipolar hypersoft set can be expressed in set-theoretic form as $(\mathcal{K}, \mathcal{L}, \mathcal{R}) = \{(s, \mathcal{K}(s), \mathcal{L}(s)) : s \in \mathcal{R} \wedge \mathcal{K}(s) \cap \mathcal{L}(s) = \emptyset\}$.

3 Bipolar Spherical Fuzzy Hypersoft Sets

In this section, we examined the fundamental notion of the bipolar spherical fuzzy hypersoft set, which emerges from the integration of spherical fuzzy theory with the framework of bipolar hypersoft sets. This hybrid structure is designed to represent complex decision information by simultaneously incorporating positive and negative assessments under a multi-attribute parameterization scheme.

Definition 3.1. Let $\mathbf{L}_{\mathbb{k}}$ be a universe of discourse and $\mathbb{P}(\mathbf{L}_{\mathbb{k}})$ denote its power set. Let $\mathbf{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \dots, \mathbf{c}_n\}$ be a finite collection of mutually disjoint parameters whose corresponding attribute value sets are $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \dots, \mathcal{R}_n$, respectively and $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2 \times \dots \times \mathcal{R}_n$. A bipolar spherical fuzzy hypersoft set (BSFHSS) over $\mathbf{L}_{\mathbb{k}}$ is defined as a triplet $(\mathcal{B}, \mathcal{H}, \mathcal{R})$, where $\mathcal{B} : \mathcal{R} \rightarrow SF^+(\mathbf{L}_{\mathbb{k}})$ and $\mathcal{H} : \mathcal{R} \rightarrow SF^-(\mathbf{L}_{\mathbb{k}})$ are mappings assigning positive and negative spherical fuzzy information, respectively. For each $s \in \mathcal{R}$ and $\mathbf{l}_{\mathbb{k}} \in \mathbf{L}_{\mathbb{k}}$, $\mathcal{B}(s) = \langle \mu^+(\mathbf{l}_{\mathbb{k}}), \eta^+(\mathbf{l}_{\mathbb{k}}), \nu^+(\mathbf{l}_{\mathbb{k}}) \rangle$, $\mathcal{H}(s) = \langle \mu^-(\mathbf{l}_{\mathbb{k}}), \eta^-(\mathbf{l}_{\mathbb{k}}), \nu^-(\mathbf{l}_{\mathbb{k}}) \rangle$, such that $(\mu^+)^2 + (\eta^+)^2 + (\nu^+)^2 \leq 1$, $(\mu^-)^2 + (\eta^-)^2 + (\nu^-)^2 \leq 1$, where $\mu^+, \eta^+, \nu^+ \in [0, 1]$ and $\mu^-, \eta^-, \nu^- \in [-1, 0]$.

Moreover, the positive and negative spherical fuzzy images are disjoint, i.e., $\mathcal{B}(s) \cap \mathcal{H}(s) = \emptyset$, $\forall s \in \mathcal{R}$. The BSFHSS can be represented as

$$(\mathcal{B}, \mathcal{H}, \mathcal{R}) = \{(s, \mathcal{B}(s), \mathcal{H}(s)) : s \in \mathcal{R}\}.$$

Definition 3.2. Let $(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)$ and $(\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$ be two BSFHSSs over the universe of discourse $\mathbf{L}_{\mathbb{k}}$. Then $(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)$ is said to be a BSFHSS subset of $(\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$, denoted by $(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1) \subseteq (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$, if and only if the following conditions hold:

- (i) $\mathcal{R}_1 \subseteq \mathcal{R}_2$,

(ii) $\mathcal{B}_1(s) \subseteq \mathcal{B}_2(s)$ and $\mathcal{H}_1(s) \subseteq \mathcal{H}_2(s)$, for all $s \in \mathcal{R}_1$.

Equivalently, for every $s \in \mathcal{R}_1$ and $\mathbf{l}_k \in \mathbf{L}_k$, $\mu_1^+(\mathbf{l}_k) \leq \mu_2^+(\mathbf{l}_k)$, $\eta_1^+(\mathbf{l}_k) \leq \eta_2^+(\mathbf{l}_k)$, $\nu_1^+(\mathbf{l}_k) \geq \nu_2^+(\mathbf{l}_k)$, and $\mu_1^-(\mathbf{l}_k) \geq \mu_2^-(\mathbf{l}_k)$, $\eta_1^-(\mathbf{l}_k) \geq \eta_2^-(\mathbf{l}_k)$, $\nu_1^-(\mathbf{l}_k) \leq \nu_2^-(\mathbf{l}_k)$, since $\mu^-, \eta^-, \nu^- \in [-1, 0]$.

Example 3.1. Let $\mathbf{L}_k = \{\mathbf{l}_{k1}, \mathbf{l}_{k2}, \mathbf{l}_{k3}, \mathbf{l}_{k4}\}$ be the universe of discourse.

Let $\mathcal{R}_1 = \{s_1\} \subseteq \mathcal{R}_2 = \{s_1, s_2\}$ and $(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)$ and $(\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$ over $\mathbf{L}_k$ are defined as follows.

$$(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1) = \left\{ \begin{array}{l} \mathcal{B}_1(s_1) = \{\langle \mathbf{l}_{k1}, (0.4, 0.5, 0.3) \rangle, \langle \mathbf{l}_{k2}, (0.5, 0.4, 0.3) \rangle, \langle \mathbf{l}_{k3}, (0.3, 0.4, 0.4) \rangle, \\ \langle \mathbf{l}_{k4}, (0.5, 0.3, 0.2) \rangle\}, \\ \mathcal{H}_1(s_1) = \{\langle \mathbf{l}_{k1}, (-0.4, -0.3, -0.2) \rangle, \langle \mathbf{l}_{k2}, (-0.3, -0.4, -0.3) \rangle, \\ \langle \mathbf{l}_{k3}, (-0.2, -0.3, -0.4) \rangle, \langle \mathbf{l}_{k4}, (-0.3, -0.2, -0.5) \rangle\}. \end{array} \right\}$$

$$(\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2) = \left\{ \begin{array}{l} \mathcal{B}_2(s_1) = \{\langle \mathbf{l}_{k1}, (0.7, 0.5, 0.4) \rangle, \langle \mathbf{l}_{k2}, (0.6, 0.5, 0.4) \rangle, \langle \mathbf{l}_{k3}, (0.5, 0.6, 0.4) \rangle, \\ \langle \mathbf{l}_{k4}, (0.5, 0.4, 0.5) \rangle\}, \\ \mathcal{H}_2(s_1) = \{\langle \mathbf{l}_{k1}, (-0.6, -0.4, -0.3) \rangle, \langle \mathbf{l}_{k2}, (-0.5, -0.4, -0.4) \rangle, \\ \langle \mathbf{l}_{k3}, (-0.4, -0.5, -0.3) \rangle, \langle \mathbf{l}_{k4}, (-0.5, -0.3, -0.4) \rangle\}, \\ \mathcal{B}_2(s_2) = \{\langle \mathbf{l}_{k1}, (0.6, 0.4, 0.3) \rangle, \langle \mathbf{l}_{k2}, (0.5, 0.5, 0.4) \rangle, \\ \langle \mathbf{l}_{k3}, (0.4, 0.4, 0.5) \rangle, \langle \mathbf{l}_{k4}, (0.5, 0.3, 0.4) \rangle\}, \\ \mathcal{H}_2(s_2) = \{\langle \mathbf{l}_{k1}, (-0.5, -0.4, -0.3) \rangle, \langle \mathbf{l}_{k2}, (-0.4, -0.5, -0.2) \rangle, \\ \langle \mathbf{l}_{k3}, (-0.3, -0.4, -0.5) \rangle, \langle \mathbf{l}_{k4}, (-0.4, -0.3, -0.4) \rangle\}. \end{array} \right\}$$

Since $\mathcal{R}_1 \subseteq \mathcal{R}_2$, and for all $s \in \mathcal{R}_1$, $\mathcal{B}_1(s) \subseteq \mathcal{B}_2(s)$, $\mathcal{H}_1(s) \subseteq \mathcal{H}_2(s)$, we have $(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1) \subseteq (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$.

Definition 3.3. Let $(\mathcal{B}, \mathcal{H}, \mathcal{R})$ be a BSFHSS over the universe $\mathbf{L}_k$, where

$$\mathcal{B} : \mathcal{R} \rightarrow SF^+(\mathbf{L}_k), \quad \mathcal{H} : \mathcal{R} \rightarrow SF^-(\mathbf{L}_k).$$

The complement of $(\mathcal{B}, \mathcal{H}, \mathcal{R})$ is defined as

$$(\mathcal{B}, \mathcal{H}, \mathcal{R})^c = (\mathcal{B}^c, \mathcal{H}^c, \mathcal{R}),$$

Example 3.2. Suppose $(\mathcal{B}, \mathcal{H}, \mathcal{R})$ is a BSFHSS over the universe $\mathbf{L}_k = \{\mathbf{l}_{k1}, \mathbf{l}_{k2}, \mathbf{l}_{k3}, \mathbf{l}_{k4}\}$, with

$$(\mathcal{B}, \mathcal{H}, \mathcal{R}) = \left\{ \begin{array}{l} \mathcal{B}(s) = \{\langle \mathbf{l}_{k1}, (0.6, 0.5, 0.4) \rangle, \langle \mathbf{l}_{k2}, (0.7, 0.4, 0.5) \rangle, \\ \langle \mathbf{l}_{k3}, (0.5, 0.6, 0.4) \rangle, \langle \mathbf{l}_{k4}, (0.6, 0.5, 0.5) \rangle\}, \\ \mathcal{H}(s) = \{\langle \mathbf{l}_{k1}, (-0.5, -0.3, -0.4) \rangle, \langle \mathbf{l}_{k2}, (-0.4, -0.5, -0.3) \rangle, \\ \langle \mathbf{l}_{k3}, (-0.3, -0.4, -0.5) \rangle, \langle \mathbf{l}_{k4}, (-0.5, -0.4, -0.3) \rangle\}. \end{array} \right\}$$

The complement of $(\mathcal{B}, \mathcal{H}, \mathcal{R})$ is

$$(\mathcal{B}, \mathcal{H}, \mathcal{R})^c = \left\{ \begin{array}{l} \mathcal{B}^c(s) = \{\langle \mathbf{l}_{k1}, (0.4, 0.5, 0.6) \rangle, \langle \mathbf{l}_{k2}, (0.5, 0.4, 0.7) \rangle, \\ \langle \mathbf{l}_{k3}, (0.4, 0.6, 0.5) \rangle, \langle \mathbf{l}_{k4}, (0.5, 0.5, 0.6) \rangle\}, \\ \mathcal{H}^c(s) = \mathcal{B}(s) = \{\langle \mathbf{l}_{k1}, (-0.4, -0.3, -0.5) \rangle, \langle \mathbf{l}_{k2}, (-0.3, -0.5, -0.4) \rangle, \\ \langle \mathbf{l}_{k3}, (-0.5, -0.4, -0.3) \rangle, \langle \mathbf{l}_{k4}, (-0.3, -0.4, -0.5) \rangle\} \end{array} \right\}$$

Definition 3.4. The extended union of $(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)$ and $(\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$ defined as

$(\mathcal{B}_3, \mathcal{H}_3, \mathcal{R}_3) = (\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1) \cup_e (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$, where $\mathcal{R}_3 = \mathcal{R}_1 \cup \mathcal{R}_2$ and for all $s \in \mathcal{R}_3$,

$$\mathcal{B}_3(s) = \begin{cases} \mathcal{B}_1(s), & \text{if } s \in \mathcal{R}_1 - \mathcal{R}_2 \\ \mathcal{B}_2(s), & \text{if } s \in \mathcal{R}_2 - \mathcal{R}_1 \\ \mathcal{B}_1(s) \cup \mathcal{B}_2(s), & \text{if } s \in \mathcal{R}_1 \cap \mathcal{R}_2 \end{cases}$$

$$\mathcal{H}_3(s) = \begin{cases} \mathcal{H}_1(s), & \text{if } s \in \mathcal{R}_1 - \mathcal{R}_2 \\ \mathcal{H}_2(s), & \text{if } s \in \mathcal{R}_2 - \mathcal{R}_1 \\ \mathcal{H}_1(s) \cap \mathcal{H}_2(s) & \text{if } s \in \mathcal{R}_1 \cap \mathcal{R}_2 \end{cases}$$

Example 3.3. Let two BSFHSs $(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)$ and $(\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$ over $\mathbf{L}_k$ are defined in example 3.1 then

$$(\mathcal{B}_3, \mathcal{H}_3, \mathcal{R}_3) = \left\{ \begin{array}{l} \mathcal{B}_3(s_1) = \{\langle \mathbf{l}_{k1}, (0.7, 0.5, 0.4) \rangle, \langle \mathbf{l}_{k2}, (0.6, 0.5, 0.4) \rangle, \\ \langle \mathbf{l}_{k3}, (0.5, 0.6, 0.4) \rangle, \langle \mathbf{l}_{k4}, (0.5, 0.4, 0.5) \rangle\}, \\ \mathcal{H}_3(s_1) = \{\langle \mathbf{l}_{k1}, (-0.6, -0.4, -0.3) \rangle, \langle \mathbf{l}_{k2}, (-0.5, -0.4, -0.4) \rangle, \\ \langle \mathbf{l}_{k3}, (-0.4, -0.5, -0.4) \rangle, \langle \mathbf{l}_{k4}, (-0.5, -0.3, -0.5) \rangle\}. \\ \mathcal{B}_3(s_2) = \{\langle \mathbf{l}_{k1}, (0.6, 0.4, 0.3) \rangle, \langle \mathbf{l}_{k2}, (0.5, 0.5, 0.4) \rangle, \\ \langle \mathbf{l}_{k3}, (0.4, 0.4, 0.5) \rangle, \langle \mathbf{l}_{k4}, (0.5, 0.3, 0.4) \rangle\}, \\ \mathcal{H}_3(s_2) = \{\langle \mathbf{l}_{k1}, (-0.5, -0.4, -0.3) \rangle, \langle \mathbf{l}_{k2}, (-0.4, -0.5, -0.2) \rangle, \\ \langle \mathbf{l}_{k3}, (-0.3, -0.4, -0.5) \rangle, \langle \mathbf{l}_{k4}, (-0.4, -0.3, -0.4) \rangle\}. \end{array} \right\}$$

Definition 3.5. The extended intersection of $(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)$ and $(\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$ defined as:

$(\mathcal{B}_4, \mathcal{H}_4, \mathcal{R}_4) = (\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1) \cap_e (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$, where $\mathcal{R}_4 = \mathcal{R}_1 \cup \mathcal{R}_2$ and for all $s \in \mathcal{R}_4$,

$$\mathcal{B}_4(s) = \begin{cases} \mathcal{B}_1(s), & \text{if } s \in \mathcal{R}_1 - \mathcal{R}_2 \\ \mathcal{B}_2(s), & \text{if } s \in \mathcal{R}_2 - \mathcal{R}_1 \\ \mathcal{B}_1(s) \cap \mathcal{B}_2(s) & \text{if } s \in \mathcal{R}_1 \cap \mathcal{R}_2 \end{cases}$$

$$\mathcal{H}_4(s) = \begin{cases} \mathcal{H}_1(s), & \text{if } s \in \mathcal{R}_1 - \mathcal{R}_2 \\ \mathcal{H}_2(s), & \text{if } s \in \mathcal{R}_2 - \mathcal{R}_1 \\ \mathcal{H}_1(s) \cup \mathcal{H}_2(s) & \text{if } s \in \mathcal{R}_1 \cap \mathcal{R}_2 \end{cases}$$

Example 3.4. Let two BSFHSs $(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)$ and $(\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$ over $\mathbf{L}_k$ are defined in example 3.1 then

$$(\mathcal{B}_4, \mathcal{H}_4, \mathcal{R}_4) = \left\{ \begin{array}{l} \mathcal{B}_4(s_1) = \{ \langle \mathbf{l}_{k1}, (0.4, 0.5, 0.3) \rangle, \langle \mathbf{l}_{k2}, (0.5, 0.4, 0.3) \rangle, \\ \langle \mathbf{l}_{k3}, (0.3, 0.4, 0.4) \rangle, \langle \mathbf{l}_{k4}, (0.5, 0.3, 0.2) \rangle \}, \\ \mathcal{H}_4(s_1) = \{ \langle \mathbf{l}_{k1}, (-0.4, -0.3, -0.2) \rangle, \langle \mathbf{l}_{k2}, (-0.3, -0.4, -0.3) \rangle, \\ \langle \mathbf{l}_{k3}, (-0.2, -0.3, -0.4) \rangle, \langle \mathbf{l}_{k4}, (-0.3, -0.2, -0.4) \rangle \}, \\ \mathcal{B}_4(s_2) = \{ \langle \mathbf{l}_{k1}, (0.6, 0.4, 0.3) \rangle, \langle \mathbf{l}_{k2}, (0.5, 0.5, 0.4) \rangle, \\ \langle \mathbf{l}_{k3}, (0.4, 0.4, 0.5) \rangle, \langle \mathbf{l}_{k4}, (0.5, 0.3, 0.4) \rangle \}, \\ \mathcal{H}_4(s_2) = \{ \langle \mathbf{l}_{k1}, (-0.5, -0.4, -0.3) \rangle, \langle \mathbf{l}_{k2}, (-0.4, -0.5, -0.2) \rangle, \\ \langle \mathbf{l}_{k3}, (-0.3, -0.4, -0.5) \rangle, \langle \mathbf{l}_{k4}, (-0.4, -0.3, -0.4) \rangle \}. \end{array} \right\}$$

Definition 3.6. The OR- operation of $(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)$ and $(\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$ defined as $(\mathcal{B}_5, \mathcal{H}_5, \mathcal{R}_5) = (\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1) \vee (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$, where

$\mathcal{R}_5 = \mathcal{R}_1 \times \mathcal{R}_2$ and for all $(s_1, s_2) \in \mathcal{R}_5$ such that $s_1 \in \mathcal{R}_1$ and $s_2 \in \mathcal{R}_2$.
 $\mathcal{B}_5(s_1, s_2) = \mathcal{B}_1(s_1) \cup \mathcal{B}_2(s_2)$ and $\mathcal{H}_5(s_1, s_2) = \mathcal{H}_1(s_1) \cap \mathcal{H}_2(s_2)$

Example 3.5. Let two BSFHSs $(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)$ and $(\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$ over $\mathbf{L}_k$ are defined in example 3.1 then

$$(\mathcal{B}_5, \mathcal{H}_5, \mathcal{R}_5) = \left\{ \begin{array}{l} \mathcal{B}_5(s_1, s_1) = \{ \langle \mathbf{l}_{k1}, (0.7, 0.5, 0.4) \rangle, \langle \mathbf{l}_{k2}, (0.6, 0.5, 0.4) \rangle, \\ \langle \mathbf{l}_{k3}, (0.5, 0.6, 0.4) \rangle, \langle \mathbf{l}_{k4}, (0.5, 0.4, 0.5) \rangle \}, \\ \mathcal{H}_5(s_1, s_1) = \{ \langle \mathbf{l}_{k1}, (-0.6, -0.4, -0.3) \rangle, \langle \mathbf{l}_{k2}, (-0.5, -0.4, -0.4) \rangle, \\ \langle \mathbf{l}_{k3}, (-0.4, -0.5, -0.4) \rangle, \langle \mathbf{l}_{k4}, (-0.5, -0.3, -0.5) \rangle \}, \\ \mathcal{B}_5(s_1, s_2) = \{ \langle \mathbf{l}_{k1}, (0.7, 0.5, 0.4) \rangle, \langle \mathbf{l}_{k2}, (0.6, 0.5, 0.4) \rangle, \\ \langle \mathbf{l}_{k3}, (0.5, 0.6, 0.5) \rangle, \langle \mathbf{l}_{k4}, (0.5, 0.4, 0.5) \rangle \}, \\ \mathcal{H}_5(s_1, s_2) = \{ \langle \mathbf{l}_{k1}, (-0.5, -0.4, -0.3) \rangle, \langle \mathbf{l}_{k2}, (-0.4, -0.5, -0.3) \rangle, \\ \langle \mathbf{l}_{k3}, (-0.3, -0.4, -0.4) \rangle, \langle \mathbf{l}_{k4}, (-0.4, -0.3, -0.4) \rangle \}. \end{array} \right\}$$

Definition 3.7. The AND- operation of $(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)$ and $(\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$ defined as:

$(\mathcal{B}_6, \mathcal{H}_6, \mathcal{R}_6) = (\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1) \wedge (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$, where $\mathcal{R}_6 = \mathcal{R}_1 \times \mathcal{R}_2$ and for all $(s_1, s_2) \in \mathcal{R}_6$ such that $s_1 \in \mathcal{R}_1$ and $s_2 \in \mathcal{R}_2$.

$\mathcal{B}_6(s_1, s_2) = \mathcal{B}_1(s_1) \cap \mathcal{B}_2(s_2)$ and $\mathcal{H}_6(s_1, s_2) = \mathcal{H}_1(s_1) \cup \mathcal{H}_2(s_2)$

Example 3.6. Let two BSFHSs $(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)$ and $(\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$ over $\mathbf{L}_k$ are

defined in example 3.1 then

$$(\mathcal{B}_6, \mathcal{H}_6, \mathcal{R}_6) = \left\{ \begin{array}{l} \mathcal{B}_6(s_1, s_1) = \{ \langle \mathbf{l}_{k1}, (0.4, 0.5, 0.4) \rangle, \langle \mathbf{l}_{k2}, (0.5, 0.4, 0.4) \rangle, \\ \langle \mathbf{l}_{k3}, 0.3, 0.4, 0.4 \rangle, \langle \mathbf{l}_{k4}, (0.5, 0.3, 0.5) \rangle \}, \\ \mathcal{H}_6(s_1, s_1) = \{ \langle \mathbf{l}_{k1}, (-0.4, -0.3, -0.3) \rangle, \langle \mathbf{l}_{k2}, (-0.3, -0.4, -0.4) \rangle, \\ \langle \mathbf{l}_{k3}, (-0.2, -0.3, -0.4) \rangle, \langle \mathbf{l}_{k4}, (-0.3, -0.2, -0.5) \rangle \}, \\ \mathcal{B}_6(s_1, s_2) = \{ \langle \mathbf{l}_{k1}, (0.4, 0.4, 0.3) \rangle, \langle \mathbf{l}_{k2}, (0.5, 0.4, 0.4) \rangle, \\ \langle \mathbf{l}_{k3}, (0.3, 0.4, 0.5) \rangle, \langle \mathbf{l}_{k4}, (0.5, 0.3, 0.4) \rangle \}, \\ \mathcal{H}_6(s_1, s_2) = \{ \langle \mathbf{l}_{k1}, (-0.4, -0.3, -0.3) \rangle, \langle \mathbf{l}_{k2}, (-0.3, -0.4, -0.2) \rangle, \\ \langle \mathbf{l}_{k3}, (-0.2, -0.3, -0.5) \rangle, \langle \mathbf{l}_{k4}, (-0.3, -0.2, -0.5) \rangle \}. \end{array} \right\}$$

3.1 De Morgan's laws for BSFHSS

In this section, the extended union and extended intersection of BSFHSSs are examined. De Morgan's laws play a dominant role in establishing the dual relationship between these operations. It is demonstrated that the defined complement operator ensures the validity of De Morgan's laws within the BSFHSS framework.

Theorem 3.1. *If $(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)$ and $(\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$ be two BPFHSS over $\mathbf{L}_k$. then*

- (i) $((\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1) \cup_e (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2))^c = (\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)^c \cap_e (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)^c$
- (ii) $((\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1) \cap_e (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2))^c = (\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)^c \cup_e (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)^c$

Proof. (i) Since
 $(\mathcal{B}_3, \mathcal{H}_3, \mathcal{R}_3) = (\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1) \cup_e (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)$,
 where $\mathcal{R}_3 = \mathcal{R}_1 \cup \mathcal{R}_2$

$$\mathcal{B}_3(s) = \left\{ \begin{array}{ll} \mathcal{B}_1(s), & \text{if } s \in \mathcal{R}_1 - \mathcal{R}_2 \\ \mathcal{B}_2(s), & \text{if } s \in \mathcal{R}_2 - \mathcal{R}_1 \\ \mathcal{B}_1 \cup \mathcal{B}_2(s) & \text{if } s \in \mathcal{R}_1 \cap \mathcal{R}_2 \end{array} \right\}$$

$$\mathcal{H}_3(s) = \left\{ \begin{array}{ll} \mathcal{H}_1(s), & \text{if } s \in \mathcal{R}_1 - \mathcal{R}_2 \\ \mathcal{H}_2(s), & \text{if } s \in \mathcal{R}_2 - \mathcal{R}_1 \\ \mathcal{H}_1(s) \cap \mathcal{H}_2(s) & \text{if } s \in \mathcal{R}_1 \cap \mathcal{R}_2 \end{array} \right\}$$

for all $s \in \mathcal{R}_3$ then

$$(\mathcal{B}_3, \mathcal{H}_3, \mathcal{R}_3)^c = ((\mathcal{B}_1, \mathcal{H}_1) \cup_e (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2))^c =$$

$$(\mathcal{B}_3(s))^c = \left\{ \begin{array}{ll} \mathcal{H}_1(s), & \text{if } s \in \mathcal{R}_1 - \mathcal{R}_2 \\ \mathcal{H}_2(s), & \text{if } s \in \mathcal{R}_2 - \mathcal{R}_1 \\ \mathcal{H}_1(s) \cap \mathcal{H}_2(s) & \text{if } s \in \mathcal{R}_1 \cap -\mathcal{R}_2 \end{array} \right\}$$

$$(\mathcal{H}_3(s))^c = \left\{ \begin{array}{ll} \mathcal{B}_1(s) & \text{if } s \in \mathcal{R}_1 - \mathcal{R}_2 \\ \mathcal{B}_2(s) & \text{if } s \in \mathcal{R}_2 - \mathcal{R}_1 \\ \mathcal{B}_1(s) \cup \mathcal{B}_2(s) & \text{if } s \in \mathcal{R}_1 \cap \mathcal{R}_2 \end{array} \right\}$$

$$(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)^c \cap_e (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)^c =$$

$$(\mathcal{B}_3(s))^c = \left\{ \begin{array}{ll} \mathcal{H}_1(s), & \text{if } s \in \mathcal{R}_1 - \mathcal{R}_2 \\ \mathcal{H}_2(s), & \text{if } s \in \mathcal{R}_2 - \mathcal{R}_1 \\ \mathcal{H}_1(s) \cap \mathcal{H}_2(s) & \text{if } s \in \mathcal{R}_1 \cap -\mathcal{R}_2 \end{array} \right\}$$

$$(\mathcal{H}_3(s))^c = \left\{ \begin{array}{ll} \mathcal{B}_1(s) & \text{if } s \in \mathcal{R}_1 - \mathcal{R}_2 \\ \mathcal{B}_2(s) & \text{if } s \in \mathcal{R}_2 - \mathcal{R}_1 \\ \mathcal{B}_1(s) \cup \mathcal{B}_2(s) & \text{if } s \in \mathcal{R}_1 \cap \mathcal{R}_2 \end{array} \right\}$$

$$\text{Hence } ((\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1) \cup_e (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2))^c =$$

$$(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)^c \cap_e (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)^c$$

(ii) Since

$$(\mathcal{B}_3, \mathcal{H}_3, \mathcal{R}_3) = (\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1) \cap_e (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2),$$

where $\mathcal{R}_3 = \mathcal{R}_1 \cup \mathcal{R}_2$

$$\mathcal{B}_3(s) = \left\{ \begin{array}{ll} \mathcal{B}_1(s), & \text{if } s \in \mathcal{R}_1 - \mathcal{R}_2 \\ \mathcal{B}_2(s), & \text{if } s \in \mathcal{R}_2 - \mathcal{R}_1 \\ \mathcal{B}_1(s) \cap \mathcal{B}_2(s) & \text{if } s \in \mathcal{R}_1 \cap \mathcal{R}_2 \end{array} \right\}$$

$$\mathcal{H}_3(s) = \left\{ \begin{array}{ll} \mathcal{H}_1(s), & \text{if } s \in \mathcal{R}_1 - \mathcal{R}_2 \\ \mathcal{H}_2(s), & \text{if } s \in \mathcal{R}_2 - \mathcal{R}_1 \\ \mathcal{H}_1(s) \cup \mathcal{H}_2(s) & \text{if } s \in \mathcal{R}_1 \cap -\mathcal{R}_2 \end{array} \right\}$$

$$\text{then } ((\mathcal{B}_3, \mathcal{H}_3, \mathcal{R}_3))^c =$$

$$((\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1) \cap_e (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2))^c =$$

$$(\mathcal{B}_3(s))^c = \left\{ \begin{array}{ll} \mathcal{H}_1(s), & \text{if } s \in \mathcal{R}_1 - \mathcal{R}_2 \\ \mathcal{H}_2(s), & \text{if } s \in \mathcal{R}_2 - \mathcal{R}_1 \\ \mathcal{H}_1(s) \cup \mathcal{H}_2(s) & \text{if } s \in \mathcal{R}_1 \cap -\mathcal{R}_2 \end{array} \right\}$$

$$(\mathcal{H}_3(s))^c = \left\{ \begin{array}{ll} \mathcal{B}_1(s) & \text{if } s \in \mathcal{R}_1 - \mathcal{R}_2 \\ \mathcal{B}_2(s) & \text{if } s \in \mathcal{R}_2 - \mathcal{R}_1 \\ \mathcal{B}_1(s) \cap \mathcal{B}_2(s) & \text{if } s \in \mathcal{R}_1 \cap \mathcal{R}_2 \end{array} \right\}$$

$$(\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)^c \cup_e (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)^c =$$

$$(\mathcal{B}_3(s))^c = \left\{ \begin{array}{ll} \mathcal{H}_1(s), & \text{if } s \in \mathcal{R}_1 - \mathcal{R}_2 \\ \mathcal{H}_2(s), & \text{if } s \in \mathcal{R}_2 - \mathcal{R}_1 \\ \mathcal{H}_1(s) \cup \mathcal{H}_2(s) & \text{if } s \in \mathcal{R}_1 \cap -\mathcal{R}_2 \end{array} \right\}$$

$$(\mathcal{H}_3(s))^c = \left\{ \begin{array}{ll} \mathcal{B}_1(s) & \text{if } s \in \mathcal{R}_1 - \mathcal{R}_2 \\ \mathcal{B}_2(s) & \text{if } s \in \mathcal{R}_2 - \mathcal{R}_1 \\ \mathcal{B}_1(s) \cap \mathcal{B}_2(s) & \text{if } s \in \mathcal{R}_1 \cap \mathcal{R}_2 \end{array} \right\}$$

$$\text{Hence } ((\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1) \cap_e (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2))^c = (\mathcal{B}_1, \mathcal{H}_1, \mathcal{R}_1)^c \cup_e (\mathcal{B}_2, \mathcal{H}_2, \mathcal{R}_2)^c$$

4 Some Bipolar Spherical Fuzzy Hypersoft Geometric Operators

This section focuses on defining the score, certainty, and accuracy functions for bipolar spherical fuzzy hypersoft numbers (BSFHSNs).

Definition 4.1. Let

$$\mathcal{A}(s_i) = ([\mathbf{l}_{ki}, \langle \mu^+, \eta^+, \nu^+ \rangle], [\mathbf{l}_{ki}, \langle \mu^-, \eta^-, \nu^- \rangle])$$

be a BSFHSNs, where

$$(\mu^+)^2 + (\eta^+)^2 + (\nu^+)^2 \leq 1, \quad (\mu^-)^2 + (\eta^-)^2 + (\nu^-)^2 \leq 1,$$

with

$$\mu^+, \eta^+, \nu^+ \in [0, 1], \quad \mu^-, \eta^-, \nu^- \in [-1, 0].$$

Then the score, accuracy, and certainty functions of the BSFHSNs $\mathcal{A}$ are defined as follows:

(i) Score function is defined as

$$\Psi(\mathcal{A}) = \frac{1}{2} \left(\mu^+ - \eta^+ - \nu^+ + \mu^- - \eta^- - \nu^- \right),$$

(ii) Accuracy function is defined as

$$\Omega(\mathcal{A}) = \frac{1}{2}(\mu^+ + \nu^+ - \mu^- - \nu^-),$$

(iii) Certainty function is defined as

$$\Phi(\mathcal{A}) = \frac{1}{2}(\mu^+ - \nu^-).$$

The ranges of these functions are given by

$$\Psi(\mathcal{A}) \in [-1, 1], \quad \Omega(\mathcal{A}) \in [0, 1], \quad \Phi(\mathcal{A}) \in [0, 1].$$

Definition 4.2. Let

$$\mathbb{B}_{ij} = (\mu_{ij}^+, \eta_{ij}^+, \nu_{ij}^+, \mu_{ij}^-, \eta_{ij}^-, \nu_{ij}^-), \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m,$$

be a family of bipolar spherical fuzzy hypersoft numbers (BSFHSNs), where

$$(\mu_{ij}^+)^2 + (\eta_{ij}^+)^2 + (\nu_{ij}^+)^2 \leq 1, \quad (\mu_{ij}^-)^2 + (\eta_{ij}^-)^2 + (\nu_{ij}^-)^2 \leq 1,$$

$$\mu_{ij}^+, \eta_{ij}^+, \nu_{ij}^+ \in [0, 1], \quad \mu_{ij}^-, \eta_{ij}^-, \nu_{ij}^- \in [-1, 0].$$

Let

$$\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$$

be the weight vector of decision makers (DMs) and

$$\mathbf{f} = (f_1, f_2, \dots, f_m)^T$$

be the weight vector of sub-attributes, satisfying

$$\sum_{i=1}^n \omega_i = 1, \quad \sum_{j=1}^m f_j = 1, \quad \omega_i, f_j \geq 0.$$

Then, the bipolar spherical fuzzy hypersoft weighted geometric aggregation operator ($BSFHS^{WG}_{AO}$) is defined as a mapping

$$BSFHS^{WG}_{AO} : \mathbb{B}^{n \times m} \rightarrow \mathbb{B},$$

such that

$$BSFHS^{WG}_{AO}(\mathbb{B}_{11}, \mathbb{B}_{12}, \dots, \mathbb{B}_{nm}) = \bigotimes_{j=1}^m \left(\bigotimes_{i=1}^n \mathbb{B}_{ij}^{\omega_i} \right)^{f_j}.$$

Theorem 4.1. *Let*

$$\mathbb{B}_{ij} = (\mu_{ij}^+, \eta_{ij}^+, \nu_{ij}^+, \mu_{ij}^-, \eta_{ij}^-, \nu_{ij}^-), \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m,$$

be a family of bipolar spherical fuzzy hypersoft numbers (BSFHSNs). Then, the aggregation result obtained by the $BSFHS_{AO}^{WG}$ is also a BSFHSN and is given by

$$\begin{aligned} & BSFHS_{AO}^{WG}(\mathbb{B}_{11}, \mathbb{B}_{12}, \dots, \mathbb{B}_{nm}) = (\mu^+, \eta^+, \nu^+, \mu^-, \eta^-, \nu^-), \\ & = \\ & \left\{ \begin{aligned} & \prod_{j=1}^m \left(\prod_{i=1}^n (\mu_{ij}^+)^{\omega_i} \right)^{f_j}, \sqrt{1 - \prod_{j=1}^m \left(\prod_{i=1}^n (1 - (\eta_{ij}^+)^2)^{\omega_i} \right)^{f_j}}, \\ & \sqrt{1 - \prod_{j=1}^m \left(\prod_{i=1}^n (1 - (\nu_{ij}^+)^2)^{\omega_i} \right)^{f_j}}, \\ & -\sqrt{1 - \prod_{j=1}^m \left(\prod_{i=1}^n (1 - (\mu_{ij}^-)^2)^{\omega_i} \right)^{f_j}}, -\prod_{j=1}^m \left(\prod_{i=1}^n |\eta_{ij}^-|^{\omega_i} \right)^{f_j}, -\prod_{j=1}^m \left(\prod_{i=1}^n |\nu_{ij}^-|^{\omega_i} \right)^{f_j} \end{aligned} \right\}. \end{aligned}$$

Theorem 4.2. *Let*

$$\mathbb{B}_{ij} = (\mu_{ij}^+, \eta_{ij}^+, \nu_{ij}^+, \mu_{ij}^-, \eta_{ij}^-, \nu_{ij}^-), \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m,$$

be a collection of BSFHSNs and let

$$BSFHSGWG(\mathbb{B}_{11}, \mathbb{B}_{12}, \dots, \mathbb{B}_{nm}) = (\mu^+, \eta^+, \nu^+, \mu^-, \eta^-, \nu^-)$$

be their aggregated value. Then, the $BSFHS_{AO}^{WG}$ satisfies the following properties.

(i) *Idempotency*

If

$$\mathbb{B}_{ij} = \mathbb{B} = (\mu^+, \eta^+, \nu^+, \mu^-, \eta^-, \nu^-) \quad \forall i, j,$$

then

$$BSFHSGWG(\mathbb{B}, \mathbb{B}, \dots, \mathbb{B}) = \mathbb{B}.$$

Proof. *Since $\sum_{i=1}^n \omega_i = 1$ and $\sum_{j=1}^m f_j = 1$, we have*

$$\mu^+ = \prod_{j=1}^m \left(\prod_{i=1}^n (\mu^+)^{\omega_i} \right)^{f_j} = (\mu^+)^{\sum f_j} = \mu^+.$$

Similarly, the expressions of $\eta^+, \nu^+, \mu^-, \eta^-, \nu^-$ reduce to their original values. Hence, the operator is idempotent. $\square$

(ii) *Monotonicity*

Let

$$\mathbb{B}_{ij} \leq \mathbb{C}_{ij} \quad \forall i, j,$$

that is,

$$\begin{aligned} \mu_{ij}^+ &\leq \tilde{\mu}_{ij}^+, \quad \eta_{ij}^+ \geq \tilde{\eta}_{ij}^+, \quad \nu_{ij}^+ \geq \tilde{\nu}_{ij}^+, \\ \mu_{ij}^- &\geq \tilde{\mu}_{ij}^-, \quad \eta_{ij}^- \leq \tilde{\eta}_{ij}^-, \quad \nu_{ij}^- \leq \tilde{\nu}_{ij}^-. \end{aligned}$$

Then,

$$BSFHSGWG(\mathbb{B}_{ij}) \leq BSFHSGWG(\mathbb{C}_{ij}).$$

Proof. The weighted geometric mean is monotonically increasing for positive membership degrees and monotonically decreasing for neutral and non-membership degrees. Since all aggregation components preserve these orders, the BSFHSG-WG AO is monotonic. $\square$

(iii) *Boundedness*

Let

$$\mathbb{B}^- = (\min \mu_{ij}^+, \max \eta_{ij}^+, \max \nu_{ij}^+, \max \mu_{ij}^-, \min \eta_{ij}^-, \min \nu_{ij}^-),$$

and

$$\mathbb{B}^+ = (\max \mu_{ij}^+, \min \eta_{ij}^+, \min \nu_{ij}^+, \min \mu_{ij}^-, \max \eta_{ij}^-, \max \nu_{ij}^-).$$

Then,

$$\mathbb{B}^- \leq BSFHSGWG(\mathbb{B}_{ij}) \leq \mathbb{B}^+.$$

Proof. Since the geometric aggregation operator produces values lying between the minimum and maximum of the aggregated inputs, the result is bounded by $\mathbb{B}^-$ and $\mathbb{B}^+$. $\square$

5 MCDM Based on Some Bipolar Spherical Fuzzy Hypersoft Geometric Operators

Suppose that Let $\mathcal{L}_k = \{\mathbf{l}_{k1}, \mathbf{l}_{k2}, \mathbf{l}_{k3}, \dots, \mathbf{l}_{kr}\}$ be a universe of discourse and $\mathbf{P}(\mathcal{L}_k)$ its power set. Let $\mathbf{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \dots, \mathbf{c}_q\}$ be a set of n disjoint parameters whose corresponding attribute values are $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \dots, \mathcal{R}_n$ such that $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_3 \times \dots \times \mathcal{R}_q$, with $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ for $i \neq j, i, j \in \{1, 2, \dots, n\}$.

$$\mathcal{R} = \{s_1, s_2, s_3, \dots, s_q\}$$

is the set of criteria. Let

$$T = (T_1, T_2, \dots, T_q)$$

be the weight vector of criteria such that

$$T_i \in [0, 1], \quad \sum_{i=1}^q T_i = 1.$$

The alternatives are evaluated by the decision maker (DM) in the form of BSFHSNs.

Let

$$\beta = (\rho_{ij})_{r \times q}$$

be the BSFHS decision matrix, where

$$\rho_{ij} = (\mu_{ij}^+, \eta_{ij}^+, \nu_{ij}^+, \mu_{ij}^-, \eta_{ij}^-, \nu_{ij}^-)$$

denotes the assessment of alternative $\mathbf{l}_{ki}$ under criterion s_j .

The BSFHSNs satisfy the following conditions:

(i)

$$\mu_{ij}^+, \eta_{ij}^+, \nu_{ij}^+, |\mu_{ij}^-|, |\eta_{ij}^-|, |\nu_{ij}^-| \in [0, 1];$$

(ii)

$$(\mu_{ij}^+)^2 + (\eta_{ij}^+)^2 + (\nu_{ij}^+)^2 \leq 1,$$

$$(\mu_{ij}^-)^2 + (\eta_{ij}^-)^2 + (\nu_{ij}^-)^2 \leq 1.$$

Algorithm:

Step 1: Construction of BSFHS Decision Matrix,

$$\beta = \begin{pmatrix} (\mu_{11}^+, \eta_{11}^+, \nu_{11}^+, \mu_{11}^-, \eta_{11}^-, \nu_{11}^-) & (\mu_{12}^+, \eta_{12}^+, \nu_{12}^+, \mu_{12}^-, \eta_{12}^-, \nu_{12}^-) & \cdots & (\mu_{1q}^+, \eta_{1q}^+, \nu_{1q}^+, \mu_{1q}^-, \eta_{1q}^-, \nu_{1q}^-) \\ (\mu_{21}^+, \eta_{21}^+, \nu_{21}^+, \mu_{21}^-, \eta_{21}^-, \nu_{21}^-) & (\mu_{22}^+, \eta_{22}^+, \nu_{22}^+, \mu_{22}^-, \eta_{22}^-, \nu_{22}^-) & \cdots & (\mu_{2q}^+, \eta_{2q}^+, \nu_{2q}^+, \mu_{2q}^-, \eta_{2q}^-, \nu_{2q}^-) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{r1}^+, \eta_{r1}^+, \nu_{r1}^+, \mu_{r1}^-, \eta_{r1}^-, \nu_{r1}^-) & (\mu_{r2}^+, \eta_{r2}^+, \nu_{r2}^+, \mu_{r2}^-, \eta_{r2}^-, \nu_{r2}^-) & \cdots & (\mu_{rq}^+, \eta_{rq}^+, \nu_{rq}^+, \mu_{rq}^-, \eta_{rq}^-, \nu_{rq}^-) \end{pmatrix}.$$

Step 2: Normalization of the Decision Matrix

The normalized BSFHS decision matrix

$$N_\beta = (\rho_{ij})_{r \times q}$$

$$\text{is obtained as } N_\beta = \begin{pmatrix} [\rho_{ij}]^c & i = j \\ \rho_{ij} & i \neq j \end{pmatrix}$$

Step 3: Aggregation Using BSFHS Weighted Geometric Operator

The collective aggregated value of the alternative is computed as where

$$\mathbf{g}_t = \left(\mu_t^{ag+}, \eta_t^{ag+}, \nu_t^{ag+}, \mu_t^{ag-}, \eta_t^{ag-}, \nu_t^{ag-} \right)$$

$$= \left\{ \begin{array}{l} \frac{\prod_{j=1}^q \left(\prod_{i=1}^r (\mu_{ij}^+)^{\omega_i} \right)^{f_j}, \sqrt{1 - \prod_{j=1}^q \left(\prod_{i=1}^r (1 - (\eta_{ij}^+)^2)^{\omega_i} \right)^{f_j}},}{\sqrt{1 - \prod_{j=1}^q \left(\prod_{i=1}^r (1 - (\nu_{ij}^+)^2)^{\omega_i} \right)^{f_j}},} \\ -\sqrt{1 - \prod_{j=1}^q \left(\prod_{i=1}^r (1 - (\mu_{ij}^-)^2)^{\omega_i} \right)^{f_j}}, -\prod_{j=1}^q \left(\prod_{i=1}^r |\eta_{ij}^-|^{\omega_i} \right)^{f_j}, -\prod_{j=1}^q \left(\prod_{i=1}^r |\nu_{ij}^-|^{\omega_i} \right)^{f_j} \end{array} \right\}.$$

Step 4: Score Function

The score value of the aggregated BSFHSN $\mathbf{g}_t$ is defined as

$$\Psi(\mathbf{g}_t) = \frac{\mu_t^{ag+} - \eta_t^{ag+} - \nu_t^{ag+} + \mu_t^{ag-} - \eta_t^{ag-} - \nu_t^{ag-}}{2}, \quad t = 1, 2, \dots, r.$$

Step 5: Ranking of Alternatives

Rank all alternatives according to descending order of score values $\Psi(\mathbf{g}_t)$. The alternative with the highest score is considered the most desirable option.

Performance Analysis of Crops using BSFHSS

The goal of precision agriculture and agri-technology is to categorize and evaluate crops, soil conditions, irrigation methods, and farming practices to optimize productivity. Applications include crop yield prediction, soil fertility assessment, irrigation management, pest and disease detection, crop quality analysis, smart fertilizer application, and climate-adaptive farming strategies.

Performance analysis and quality control are critical in agriculture to ensure high crop yields, sustainable practices, and overall farm productivity. Evaluating the performance of crops and farming practices helps identify weak areas early, optimize resource allocation, and improve product quality for market competitiveness. Effective performance analysis in agriculture enhances productivity, reduces losses, and ensures sustainability.

In this research, the designed **BSFHSS-based algorithm** is applied for the performance analysis of crops. Crops are evaluated for their growth and productivity potential, considering environmental factors and farming practices. Several factors are considered in this analysis:

1. **Soil Fertility:** Nutrient content, pH, and organic matter impact crop growth.

2. **Irrigation Efficiency:** Proper water supply is critical. Over-irrigation or under-irrigation can affect growth.
3. **Crop Variety:** Different crop varieties have different yield potential and resistance to pests or drought.
4. **Fertilizer Application:** Type, timing, and quantity of fertilizers influence crop yield.
5. **Pest and Disease Management:** Healthy crops require effective pest and disease control.
6. **Harvest Quality:** Crop uniformity, size, weight, and quality indicators are analyzed for final yield assessment.

These parameters are used to perform a sensitivity and performance analysis of crops using the BSFHSS distance measures.

Crops and Attributes

Consider a farm with four crop alternatives, denoted by $\mathcal{L}_{kt}$ ($t = 1, 2, 3, 4$). Performance is analyzed based on the following attributes:

$$\mathbf{C} = \left\{ \begin{array}{l} \mathbf{c}_1 = \text{Soil Fertility}, \mathbf{c}_2 = \text{Water Management}, \mathbf{c}_3 = \text{Crop Growth}, \\ \mathbf{c}_4 = \text{Harvest Quality} \end{array} \right\}$$

Each attribute is further divided into sub-attributes for the hypersoft set structure:

$$\left\{ \begin{array}{l} \mathcal{R}_1 = \{b_{11} = \text{Nitrogen Content}, b_{12} = \text{pH Level}, b_{13} = \text{Organic Matter}\} \\ \mathcal{R}_2 = \{b_{21} = \text{Irrigation Frequency}, b_{22} = \text{Water Efficiency}, b_{23} = \text{Moisture Retention}\} \\ \mathcal{R}_3 = \{b_{31} = \text{Growth Rate}, b_{32} = \text{Pest Resistance}, b_{33} = \text{Drought Tolerance}\} \\ \mathcal{R}_4 = \{b_{41} = \text{Yield per Plant}, b_{42} = \text{Fruit Size or Weight}, b_{43} = \text{Color Uniformity}, \\ b_{44} = \text{Market Grade}\} \end{array} \right.$$

Hypersoft Set Structure of Crop Performance,

Experts provide their evaluation of crop performance using the BSFHSS structure. For simplicity, a step-by-step example is illustrated for four representative performance outcomes:

$$\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_3 \times \mathcal{R}_4$$

$$\mathcal{R} = \left\{ \begin{array}{l} s_1 = (b_{11}, b_{21}, b_{31}, b_{42}), \\ s_2 = (b_{12}, b_{23}, b_{33}, b_{44}), \\ s_3 = (b_{12}, b_{22}, b_{31}, b_{42}), \\ s_4 = (b_{11}, b_{23}, b_{33}, b_{43}) \end{array} \right\}.$$

Using these structures, BSFHSS-based aggregation and scoring can identify the most suitable crop alternative, considering all environmental, growth, and harvest parameters. This ensures robust and data-driven decision-making for precision agriculture.

Then BSFHSS is given as: $(\mathcal{B}, \mathcal{H}, \mathcal{R}) =$

$$\left\{ \begin{array}{l} (\mathcal{B}_1, \mathcal{H}_1)(s_1) = \left\{ (0.5, 0.4, 0.2, -0.4, -0.2, -0.1)/\mathbf{l}_{k1}, (0.6, 0.4, 0.3, -0.3, -0.2, -0.2)/\mathbf{l}_{k2}, \right. \\ \quad \left. (0.5, 0.4, 0.2, -0.3, -0.2, -0.3)/\mathbf{l}_{k3}, (0.4, 0.5, 0.3, -0.2, -0.3, -0.2)/\mathbf{l}_{k4} \right\}, \\ (\mathcal{B}_1, \mathcal{H}_1)(s_2) = \left\{ (0.5, 0.4, 0.3, -0.3, -0.2, -0.2)/\mathbf{l}_{k1}, (0.6, 0.4, 0.3, -0.4, -0.1, -0.2)/\mathbf{l}_{k2}, \right. \\ \quad \left. (0.5, 0.4, 0.2, -0.4, -0.1, -0.1)/\mathbf{l}_{k3}, (0.6, 0.3, 0.3, -0.3, -0.3, -0.2)/\mathbf{l}_{k4} \right\}, \\ (\mathcal{B}_2, \mathcal{H}_2)(s_3) = \left\{ (0.6, 0.5, 0.2, -0.3, -0.1, -0.2)/\mathbf{l}_{k1}, (0.6, 0.5, 0.3, -0.2, -0.1, -0.3)/\mathbf{l}_{k2}, \right. \\ \quad \left. (0.6, 0.5, 0.1, -0.1, -0.2, -0.3)/\mathbf{l}_{k3}, (0.5, 0.5, 0.2, -0.2, -0.1, -0.3)/\mathbf{l}_{k4} \right\}, \\ (\mathcal{B}_2, \mathcal{H}_2)(s_4) = \left\{ (0.5, 0.5, 0.2, -0.2, -0.2, -0.2)/\mathbf{l}_{k1}, (0.6, 0.5, 0.2, -0.3, -0.1, -0.2)/\mathbf{l}_{k2}, \right. \\ \quad \left. (0.5, 0.5, 0.2, -0.4, 0.0, -0.2)/\mathbf{l}_{k3}, (0.7, 0.4, 0.2, -0.3, -0.2, -0.2)/\mathbf{l}_{k4} \right\} \end{array} \right\}$$

The tabular representation of BSFHSS is shown in Table 1,
 Construct the decision matrix given by the decision maker in Table 1 based on BSFHSS information.

$$\beta = \left(\begin{array}{cccc} (0.2, 0.3, 0.1, -0.6, -0.2, -0.1) & (0.5, 0.3, 0.2, -0.4, -0.2, -0.3) & (0.3, 0.3, 0.1, -0.2, -0.3, -0.4) & (0.1, 0.4, 0.2, -0.3, -0.3, -0.3) \\ (0.2, 0.3, 0.2, -0.5, -0.1, -0.2) & (0.4, 0.3, 0.2, -0.5, -0.1, -0.2) & (0.2, 0.3, 0.1, -0.8, -0.1, -0.1) & (0.5, 0.2, 0.2, -0.4, -0.4, -0.2) \\ (0.4, 0.4, 0.1, -0.5, -0.1, -0.2) & (0.4, 0.4, 0.2, -0.1, -0.3, -0.4) & (0.4, 0.5, 0.0, -0.1, -0.2, -0.5) & (0.2, 0.5, 0.1, -0.2, -0.2, -0.4) \\ (0.3, 0.4, 0.1, -0.4, -0.1, -0.3) & (0.5, 0.4, 0.1, -0.4, -0.1, -0.3) & (0.3, 0.4, 0.1, -0.7, -0.1, -0.2) & (0.6, 0.3, 0.1, -0.4, -0.3, -0.3) \end{array} \right)$$

Table 1: Bipolar Spherical Fuzzy Hypersoft Decision Table

Criteria	Alternatives	BSFHSN $(\mu^+, \eta^+, \nu^+, \mu^-, \eta^-, \nu^-)$
s_1	$\mathbf{l}_{k1}$	(0.5, 0.4, 0.2, -0.4, -0.2, -0.1)
	$\mathbf{l}_{k2}$	(0.6, 0.4, 0.3, -0.3, -0.2, -0.2)
	$\mathbf{l}_{k3}$	(0.5, 0.4, 0.2, -0.3, -0.2, -0.3)
	$\mathbf{l}_{k4}$	(0.4, 0.5, 0.3, -0.2, -0.3, -0.2)
s_2	$\mathbf{l}_{k1}$	(0.5, 0.4, 0.3, -0.3, -0.2, -0.2)
	$\mathbf{l}_{k2}$	(0.6, 0.4, 0.3, -0.4, -0.1, -0.2)
	$\mathbf{l}_{k3}$	(0.5, 0.4, 0.2, -0.4, -0.1, -0.1)
	$\mathbf{l}_{k4}$	(0.6, 0.3, 0.3, -0.3, -0.3, -0.2)
s_3	$\mathbf{l}_{k1}$	(0.6, 0.5, 0.2, -0.3, -0.1, -0.2)
	$\mathbf{l}_{k2}$	(0.6, 0.5, 0.3, -0.2, -0.1, -0.3)
	$\mathbf{l}_{k3}$	(0.6, 0.5, 0.1, -0.1, -0.2, -0.3)
	$\mathbf{l}_{k4}$	(0.5, 0.5, 0.2, -0.2, -0.1, -0.3)
s_4	$\mathbf{l}_{k1}$	(0.5, 0.5, 0.2, -0.2, -0.2, -0.2)
	$\mathbf{l}_{k2}$	(0.6, 0.5, 0.2, -0.3, -0.1, -0.2)
	$\mathbf{l}_{k3}$	(0.5, 0.5, 0.2, -0.4, 0.0, -0.2)
	$\mathbf{l}_{k4}$	(0.7, 0.4, 0.2, -0.3, -0.2, -0.2)

Step 2.

The decision matrix is normalized using the formula given below:

$$N_{\beta} = \begin{pmatrix} [\rho_{ij}]^c & i = j \\ \rho_{ij} & i \neq j \end{pmatrix}$$

A normalized matrix is given below

$$N_\beta = \begin{pmatrix} (0.2, 0.4, 0.5, -0.1, -0.2, -0.4) & (0.5, 0.4, 0.3, -0.3, -0.2, -0.2) & (0.6, 0.5, 0.2, -0.3, -0.1, -0.2) & (0.5, 0.5, 0.2, -0.2, -0.2, -0.2) \\ (0.6, 0.4, 0.3, -0.3, -0.2, -0.2) & (0.3, 0.4, 0.6, -0.2, -0.1, -0.4) & (0.6, 0.5, 0.3, -0.2, -0.1, -0.3) & (0.6, 0.5, 0.2, -0.3, -0.1, -0.2) \\ (0.5, 0.4, 0.2, -0.3, -0.2, -0.3) & (0.5, 0.4, 0.2, -0.4, -0.1, -0.1) & (0.1, 0.5, 0.6, -0.3, -0.2, -0.1) & (0.5, 0.5, 0.2, -0.4, 0.0, -0.2) \\ (0.4, 0.5, 0.3, -0.2, -0.3, -0.2) & (0.6, 0.3, 0.3, -0.3, -0.3, -0.2) & (0.5, 0.5, 0.2, -0.2, -0.1, -0.3) & (0.2, 0.4, 0.7, -0.2, -0.2, -0.3) \end{pmatrix}$$

Step 3.

Obtain the aggregated value of each alternative $\mathcal{L}_{k_i}$ using the N_β and weight vectors

$$\omega_i = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right), \quad \mathbf{f}_i = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right).$$

For each of the alternatives, the aggregated value is presented as follows:

Table 2: Aggregated Bipolar Spherical Fuzzy Hypersoft Values

Alternative	Aggregated BSFHS value
$\mathbf{l}_{k1}$	$\mathbf{g}_1 = (0.45, 0.46, 0.38, -0.23, -0.20, -0.25)$
$\mathbf{l}_{k2}$	$\mathbf{g}_2 = (0.50, 0.45, 0.41, -0.24, -0.17, -0.27)$
$\mathbf{l}_{k3}$	$\mathbf{g}_3 = (0.37, 0.48, 0.44, -0.29, -0.13, -0.18)$
$\mathbf{l}_{k4}$	$\mathbf{g}_4 = (0.41, 0.44, 0.49, -0.22, -0.23, -0.26)$

Step 4.

The score value of an aggregated bipolar spherical fuzzy hypersoft number $\mathbf{g}_j = (\mu_j^{ag+}, \eta_j^{ag+}, \nu_j^{ag+}, \mu_j^{ag-}, \eta_j^{ag-}, \nu_j^{ag-})$ is defined as:

$$\Psi(\mathbf{g}_j) = \frac{\mu_j^{ag+} - \eta_j^{ag+} - \nu_j^{ag+} + \mu_j^{ag-} - \eta_j^{ag-} - \nu_j^{ag-}}{2}, \quad j = 1, 2, \dots, r.$$

Using the aggregated values from N_β , we compute:

$$\begin{aligned} \Psi(\mathbf{g}_1) &= \frac{0.45 - 0.46 - 0.38 + (-0.23) - (-0.20) - (-0.25)}{2} = -0.085 \\ \Psi(\mathbf{g}_2) &= \frac{0.50 - 0.45 - 0.41 + (-0.24) - (-0.17) - (-0.27)}{2} = -0.08 \\ \Psi(\mathbf{g}_3) &= \frac{0.37 - 0.48 - 0.44 + (-0.29) - (-0.13) - (-0.18)}{2} = -0.265 \\ \Psi(\mathbf{g}_4) &= \frac{0.41 - 0.44 - 0.49 + (-0.22) - (-0.23) - (-0.26)}{2} = -0.125 \end{aligned}$$

Step 5.

Since a higher score indicates a more favorable alternative, from table 3 the

Table 3: Score values of aggregated BSFHSNs

Alternative	Score $\Psi(\mathbf{g}_j)$
$\mathbf{g}_1$	-0.085
$\mathbf{g}_2$	-0.08
$\mathbf{g}_3$	-0.265
$\mathbf{g}_4$	-0.125

ranking is:

$$\mathbf{l}_{k2} > \mathbf{l}_{k1} > \mathbf{l}_{k4} > \mathbf{l}_{k3}.$$

Hence, the most preferred alternative is $\mathbf{l}_{k2}$. In conclusion, multi-criteria decision-making techniques provide a systematic and reliable framework for evaluating agricultural alternatives by simultaneously accounting for multiple interacting factors. The results of the proposed approach indicate that $\mathbf{l}_{k2}$ achieves the highest overall performance under the considered conditions. This demonstrates the effectiveness of structured computational methods in supporting accurate, transparent, and sustainable agricultural decision-making.

6 Conclusions

In this study, the Bipolar Spherical Fuzzy Hypersoft Set (BSFHSS) was introduced along with its basic operations, aggregation operators, and algebraic properties to effectively support multi-criteria decision-making (MCDM) under complex and uncertain environments. The proposed BSFHSS aggregation operators enable the systematic fusion of bipolar, spherical, and hypersoft information, while the validation of De Morgans laws ensures the logical consistency and mathematical soundness of the developed framework. The agricultural performance analysis confirms that the BSFHSS-based MCDM algorithm provides stable, discriminative, and reliable results, identifying $\mathbf{l}_{k2}$ as the most suitable alternative and demonstrating the superiority of BSFHSS over existing fuzzy hypersoft structures. Future research may extend BSFHSS aggregation operators to other MCDM methods such as TOPSIS, VIKOR, and PROMETHEE. The incorporation of adaptive or data-driven weight determination techniques is another promising direction. Additionally, further investigation of advanced algebraic properties and broader real-world applications will enhance the practical relevance of the BSFHSS framework.

References

- [1] Zadeh, L. (1965). Fuzzy sets, Information and control, 8(3), 338-353.
- [2] Atanassov, K. (1986). Intuitionistic Fuzzy Sets, Fuzzy Sets and Systems, 20, 87-96.
- [3] Cuong, B.C. (2013). Picture fuzzy sets, J. Comput. Sci. Cybern, 30, 409-420.
- [4] Mahmood, T., Kifayat, U., Khan, Q., and Jan, N. An approach toward decision making and medical diagnosis problems using the concept of spherical fuzzy sets Neural Computing and Applications, 31(11), (2019) 7041- 7053.
- [5] D. Molodtsov, Soft set theoryfirst results, Comput. Math. Appl. 37 (45) (1999) 19-31.
- [6] P.K. Maji, R. Biswas, A.R. Roy, Soft set theory, Comput. Math. Appl. 45 (45) (2003) 555-562.
- [7] M.I. Ali, F. Feng, X. Liu, W.K. Min, M. Shabir, On some new operations in soft set theory, Comput. Math. Appl. 57 (9) (2009) 1547-1553.
- [8] M.I. Ali, M. Shabir, M. Naz, Algebraic structures of soft sets associated with new operations, Comput. Math. Appl. 61 (9) (2011) 2647-2654.
- [9] H. Aktas, N. agman, Soft sets and soft groups, Inf. Sci. 177 (13) (2007) 2726-2735.
- [10] P.K. Maji, A.R. Roy, R. Biswas, Fuzzy soft sets, J. Fuzzy Math. 9 (2001) 589-602.
- [11] T. Deng, X. Wang, An object-parameter approach to predicting unknown data in incomplete fuzzy soft sets, Appl. Math. Model. 37 (6) (2013) 4139-4146.
- [12] M. Naz, M. Shabir, Fuzzy soft sets and their algebraic structures, World Appl. Sci. J. 22 (2013) 45-61.
- [13] A.R. Roy, P. Maji, A fuzzy soft set theoretic approach to decision making problems, J. Comput. Appl. Math. 203 (2) (2007) 412-418.

- [14] P.K. Maji, More on intuitionistic fuzzy soft sets, in: Rough Sets, Fuzzy Sets, Data Mining and Granular Computing: 12th International Conference, RSFDGrC 2009, Delhi, India, December 15-18, 2009, in: Proceedings, vol. 12, Springer, (2009), 231-240.
- [15] Y. Yang, C. Liang, S. Ji, T. Liu, Adjustable soft discernibility matrix based on picture fuzzy soft sets and its applications in decision making, J. Intell. Fuzzy Syst. 29 (4) (2015) 1711-1722.
- [16] Perveen PA, Fathima,. Spherical fuzzy soft sets and its applications in decision-making problems. Journal of Intelligent and Fuzzy Systems 37.6 (2019): 8237-8250.
- [17] D. Dubois, H. Prade, An introduction to bipolar representations of information and preference, Int. J. Intell. Syst. 23 (8) (2008) 866-877.
- [18] M. Shabir, M. Naz, On bipolar soft sets, arXiv preprint, arXiv:1303.1344, 2013.
- [19] F. Karaaslan, S. Karatas, A new approach to bipolar soft sets and its applications, Discrete Math. Algorithms Appl. 7 (04) (2015) 1550054.
- [20] K. Hayat, T. Mahmood, Some applications of bipolar soft set: characterizations of two isomorphic hemi-rings via BSI-h-ideals, J. Adv. Math. Comput. Sci. 13 (2) (2015) 1-21.
- [21] G., Ali, N., Lateef, M.U. Zia, A Novel Cognitive Rough Approach for Severity Analysis of Autistic Children Using Spherical Fuzzy Bipolar Soft Sets. Cogn Comput 16, (2024) 3260-3285.
- [22] F. Karaaslan, I. Ahmad, A. Ullah, Bipolar soft groups, J. Intell. Fuzzy Syst. 31 (1) (2016) 651-662.
- [23] F. Smarandache, Extension of soft set to hypersoft set, and then to plithogenic hypersoft set, Neutrosophic Sets Syst. 22 (1) (2018) 168-170.
- [24] M. Saeed, M. Ahsan, M.K. Siddique, M.R. Ahmad, A study of the fundamentals of hypersoft set theory, Infin. Study (2020).
- [25] M. Saeed, A.U. Rahman, M. Ahsan, F. Smarandache, An inclusive study on fundamentals of hypersoft set, in: Theory and Application of Hypersoft Set, vol. 1, 2021.
- [26] M. Abbas, G. Murtaza, F. Smarandache, Basic operations on hypersoft sets and hypersoft point, Infin. Study (2020).

- [27] A. Yolcu, T.Y. Ozturk, Fuzzy hypersoft sets and its application to decision-making, in: Theory and Application of Hypersoft Set, vol. 50, 2021.
- [28] A. Yolcu, F. Smarandache, T.Y. ztrk, Intuitionistic fuzzy hypersoft sets, Commun. Fac. Sci. Univ. Ank. Sr. A1 Math. Stat. 70 (1) (2021) 443-455.
- [29] M. Saeed, M.I. Harl, Fundamentals of picture fuzzy hypersoft set with application, Neutrosophic Sets Syst. 53 (1) (2023) 24.
- [30] M. Saeed, M.I. Harl, M.H. Saeed, I. Mekawy, Theoretical framework for a decision support system for micro-enterprise supermarket investment risk assessment using novel picture fuzzy hypersoft graph, PLoS ONE 18 (3) (2023) e0273642.
- [31] S.Y. Musa, B.A. Asaad, Bipolar hypersoft sets, Mathematics 9 (15) (2021) 1826.
- [32] Harl, M. I., Saeed, M., Saeed, M. H., Alharbi, T., and Alballa, T. Bipolar picture fuzzy hypersoft set-based performance analysis of abrasive textiles for enhanced quality control. Heliyon, 9(9) (2023).