Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 1 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189

http://amresearchreview.com/index.php/Journal/about

Comparative Analysis of Delta Perturbation Method and Homotopy Perturbation Method for Thin Film Flow of Third-Grade Fluid

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ABSTRACT

Thin film flow of non-Newtonian fluids has significant importance in coating processes, lubrication systems, polymer processing, food industries, chemical engineering, and other industrial applications where accurate prediction of flow and heat transfer behavior is required. In particular, third-grade fluids provide a more realistic model for complex fluids because they incorporate nonlinear stress–strain characteristics that cannot be fully described by Newtonian fluid models. This study presents a comparative analysis of the Delta Perturbation Method and the Homotopy Perturbation Method for the analytical approximation of thin film flow of an incompressible third-grade fluid under lift and drainage configurations. The governing nonlinear differential equations are derived from the continuity, momentum, and energy equations by considering suitable assumptions for steady, laminar, and incompressible flow. The effects of dynamic viscosity, fluid density, film thickness, third-grade material parameters β_2 and β_3, thermal conductivity, and inclination angle are examined on velocity and temperature profiles. The comparison between the two methods is based on convergence behavior, accuracy, closeness of approximate solutions, and physical interpretation of the obtained results. The expected findings indicate that both methods provide close approximate solutions for the velocity and temperature fields; however, the Delta Perturbation Method offers a systematic perturbation-based framework for treating nonlinear terms. The results are useful for understanding non-Newtonian thin

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film behavior in coating, lubrication, food processing, chemical engineering, and polymeric fluid applications.

Keywords— Thin film flow; Third-grade fluid; Delta Perturbation Method; Homotopy Perturbation Method; Velocity profile; Temperature distribution; Non-Newtonian fluid

Introduction

Thin film flow is an important area of fluid dynamics because it appears in many natural, industrial, and engineering processes where a very thin layer of liquid moves over a solid surface under the action of gravity, surface forces, pressure gradients, or external motion. Such flows are commonly observed in coating processes, lubrication systems, wire coating, polymer processing, food processing, chemical industries, biomedical flows, microchip fabrication, and heat-transfer devices. In these applications, the thickness of the fluid layer is usually small compared with the length of the surface, but the behavior of the film strongly affects product quality, surface uniformity, temperature distribution, and flow stability. For example, in coating and wire-drawing processes, accurate control of the film thickness and velocity field is necessary to obtain a smooth and uniform coating. Similarly, in lubrication and chemical processing, the prediction of flow rate, temperature rise, and shear effects is essential for improving efficiency and reducing mechanical losses. Therefore, the mathematical analysis of thin film flow remains a significant topic in applied mathematics and fluid mechanics.

Many practical fluids involved in industrial and biological systems do not obey Newton's law of viscosity. These fluids are generally classified as non-Newtonian fluids because their shear stress is not linearly proportional to the rate of strain. Examples include polymer solutions, paints, oils, pastes, blood, ketchup, slurry, and several biological and chemical fluids. Bird et al. (1987) emphasized that non-Newtonian fluids require more complex constitutive models because their mechanical behavior includes viscosity variation, elasticity, normal stress effects, and nonlinear deformation. Among different non-Newtonian fluid models, third-grade fluids are particularly important because they provide a more advanced representation of nonlinear stress–strain behavior than Newtonian and second-grade fluid models. While second-grade fluids can describe some normal stress effects, they are limited in representing shear-thinning and shear-thickening behavior. Third-grade fluids, on the other hand, include higher-order material parameters and can better explain the complex behavior of viscoelastic fluids. The thermodynamic and mechanical foundations of third-grade fluids were discussed by Fosdick and Rajagopal (1980), making this model useful for advanced studies of nonlinear fluid motion.

The mathematical modeling of third-grade fluid flow usually leads to nonlinear differential equations. These nonlinearities arise from the constitutive relation of the fluid, the momentum equation, and heat-transfer effects included through the energy equation. In the case of thin film flow over a vertical moving belt or an inclined plane, the governing equations are generally derived from the continuity, momentum, and energy equations under suitable assumptions such as steady, laminar, incompressible, and unidirectional flow. The uploaded thesis also studies these two configurations—lift

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Volume 4, Issue 1 (2026)

Online ISSN 3007-3197
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on a vertical moving belt and drainage on an inclined plane—using velocity profile, temperature distribution, flow rate, average velocity, and net upward flow as major physical quantities. However, due to the nonlinear nature of third-grade fluid equations, exact closed-form solutions are often difficult or impossible to obtain. As a result, researchers commonly use analytical and semi-analytical approximation techniques.

Several methods have been developed to solve nonlinear fluid flow problems, including the regular perturbation method, homotopy perturbation method, homotopy analysis method, variational iteration method, Adomian decomposition method, and numerical schemes. Siddiqui et al. (2006, 2013) investigated thin film flow problems involving non-Newtonian fluids and showed the importance of approximate methods for obtaining physically meaningful solutions. The Homotopy Perturbation Method, introduced and developed by He (2006), combines the classical perturbation idea with homotopy concepts from topology. It constructs a continuous deformation from an initial approximation to the required solution by using an embedding parameter. This makes HPM useful for nonlinear problems because it does not always require the presence of a naturally small physical parameter.

The Delta Perturbation Method is another important approximation technique associated with the work of Bender and his collaborators (Bender et al., 1989, 1992). Unlike traditional perturbation techniques, DPM introduces a delta parameter to handle nonlinear terms systematically and then expands the solution in a series form. This method has been applied to nonlinear differential equations in different branches of mathematical physics and offers a structured way to obtain approximate analytical solutions. In thin film flow problems, DPM can be used to calculate velocity and temperature profiles and to compare the resulting approximations with existing perturbation-based or homotopy-based solutions.

Although previous studies have examined thin film flow of third-grade fluids and have applied different analytical methods separately, limited attention has been given to a direct comparison between the Delta Perturbation Method and the Homotopy Perturbation Method for the same third-grade fluid thin film model. This creates a research gap regarding which method provides better convergence behavior, simpler computation, closer approximation, and stronger physical interpretation under identical governing equations and boundary conditions.

Therefore, the objective of this article is to compare DPM and HPM for thin film flow of a third-grade fluid by analyzing velocity and temperature profiles under lift and drainage configurations. The novelty of the study lies in presenting a method-based comparison using the same physical model and key parameters, including dynamic viscosity, density, film thickness, thermal conductivity, inclination angle, and third-grade material parameters. This comparative framework is expected to clarify the analytical strength and physical reliability of both methods for nonlinear non-Newtonian thin film flow problems.

Literature Review

Thin Film Flow and Its Engineering Applications

Thin film flow has remained a significant research area in fluid mechanics because of its wide applications in industrial, engineering, and biological processes. A thin film is

Online ISSN 3007-3197
Print ISSN 3007-3189

generally formed when a small layer of fluid moves over a solid or free surface under the influence of gravity, shear stress, surface tension, or external motion. Such flows are frequently encountered in coating processes, lubrication, wire coating, polymer processing, food industries, biomedical flows, chemical engineering, and heat-transfer systems. The uploaded thesis also highlights applications of thin film flow in manufactured coatings, lung tissue, surface-active fluids, microchip fabrication, condensate motion, painting, and sauce dispersion on food products, showing the broad practical importance of this phenomenon. In engineering applications, accurate prediction of film thickness, velocity distribution, flow rate, and temperature distribution is essential because these quantities directly affect product quality, mechanical efficiency, and thermal performance.

Non-Newtonian and Third-Grade Fluid Models

Many fluids used in real industrial and biological systems cannot be described accurately by Newtonian fluid theory. Newtonian fluids follow a linear relationship between shear stress and rate of strain, whereas non-Newtonian fluids show nonlinear stress–strain behavior. Bird et al. (1987) provided a foundational treatment of polymeric and non-Newtonian fluids, emphasizing that complex fluids may exhibit elasticity, shear-dependent viscosity, and normal stress effects. Third-grade fluids belong to the class of viscoelastic non-Newtonian fluids and are particularly useful for modeling fluids such as polymer solutions, paints, oils, slurries, gels, biological fluids, and suspensions. The thesis explains that third-grade fluids are more suitable than second-grade fluids for representing shear-thinning and shear-thickening behavior because third-grade models contain higher-order material parameters. Fosdick and Rajagopal (1980) developed important thermodynamic foundations for fluids of third grade, while Siddiqui and Kaloni (1987) and later researchers applied these models to nonlinear flow problems. However, the same nonlinear material properties that make third-grade fluids physically realistic also make their governing equations mathematically difficult.

Analytical and Semi-Analytical Methods for Nonlinear Fluid Flow Problems

The mathematical modeling of third-grade fluid flow usually produces nonlinear ordinary or partial differential equations. These nonlinearities arise from the constitutive equation, momentum balance, boundary conditions, and heat-transfer terms in the energy equation. Exact solutions are often unavailable, especially for thin film flow under lift and drainage conditions. Therefore, researchers have used approximate analytical and semi-analytical methods such as regular perturbation method, Homotopy Perturbation Method, Homotopy Analysis Method, Variational Iteration Method, Adomian Decomposition Method, and numerical methods. Hayat and Sajid (2007) discussed the difficulty of solving nonlinear fluid problems when complicated boundary conditions and nonlinear terms are present. The thesis also notes that nonlinear mathematical models often require approximate techniques, including HPM, HAM, VIM, OHAM, perturbation methods, and ADM. These methods are valuable because they provide closed-form or series-type solutions that allow physical interpretation of parameters such as viscosity, density, film thickness, thermal conductivity, and material constants.

Delta Perturbation Method in Nonlinear Differential Equations

The Delta Perturbation Method was developed by Bender and his collaborators in the

Online ISSN 3007-3197
Print ISSN 3007-3189

late 1980s as a modified perturbation technique for nonlinear problems. Bender et al. (1989, 1992) used the delta expansion to handle nonlinearities in mathematical physics and quantum field theory. Unlike classical perturbation methods, DPM introduces an artificial delta parameter into the nonlinear term and expands the solution in powers of this parameter. This approach allows the nonlinear structure of the governing equation to be treated systematically. In the uploaded thesis, DPM is applied to thin film flow of a third-grade fluid for both lift over a vertical moving belt and drainage over an inclined plane. The method is used to obtain velocity profiles, temperature distributions, flow rate, average velocity, and net upward flow. The thesis further reports that DPM results are very close to existing perturbation-method solutions, with small absolute differences in the comparison tables. This supports the usefulness of DPM as an analytical approximation tool for nonlinear fluid flow.

Homotopy Perturbation Method in Fluid Dynamics

The Homotopy Perturbation Method, introduced by He (2006), combines classical perturbation theory with homotopy concepts. In HPM, an embedding parameter is introduced to continuously deform an initial approximation into the desired solution. One major advantage of HPM is that it does not always require a naturally small physical parameter, which makes it attractive for nonlinear fluid dynamics problems. HPM has been widely applied to boundary-layer flows, heat transfer, thin film flows, non-Newtonian fluids, and nonlinear differential equations. Compared with traditional perturbation methods, HPM often gives rapidly convergent series solutions with fewer computational steps. However, its accuracy depends strongly on the choice of initial approximation, construction of the homotopy, and treatment of boundary conditions. For third-grade fluid problems, this is especially important because the nonlinear material parameters can strongly affect the velocity and temperature fields.

Research Gap Leading to the Present Comparative Study

Previous studies have investigated third-grade fluid flow using perturbation, homotopy-based, decomposition, and numerical techniques. Siddiqui et al. (2006, 2013) contributed important work on thin film flow and drainage/withdrawal problems for non-Newtonian fluids, while Zahid et al. (2017) and Bhatti et al. (2019) further extended third-grade fluid analysis to related coating and industrial flow situations. However, most studies apply one analytical method at a time and do not provide a direct comparison between DPM and HPM for the same third-grade thin film flow model. The uploaded thesis provides a strong base by solving lift and drainage problems through DPM and comparing DPM with an existing perturbation solution. The present study extends this direction by comparing Delta Perturbation Method and Homotopy Perturbation Method in terms of convergence, accuracy, closeness of approximate solutions, and physical interpretation. This comparative approach is important because it can clarify which method gives a more reliable and mathematically efficient approximation for nonlinear thin film flow of third-grade fluids under identical governing equations and boundary conditions.

Mathematical Formulation

The present analysis considers the thin film flow of an incompressible third-grade fluid under two standard configurations: lift flow over a vertical moving belt and drainage

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Volume 4, Issue 1 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189
http://amresearchreview.com/index.php/Journal/about

flow over an inclined plane. These configurations are consistent with the thesis model, where the flow is treated as steady, laminar, incompressible, and unidirectional, and the velocity and temperature fields are obtained from the continuity, momentum, and energy equations. The same mathematical model provides the basis for comparing the Delta Perturbation Method and Homotopy Perturbation Method.

Let a Cartesian coordinate system be selected such that the coordinate normal to the solid surface is denoted by y, while the flow direction is taken along x. For the lift problem, the belt moves vertically upward with constant velocity U, while the fluid film of uniform thickness $h$is carried along the belt. For the drainage problem, the fluid flows down an inclined plane under the action of gravity, where αdenotes the angle of inclination. The velocity field and temperature field are assumed in the form

$$\mathbf{V} = [u(y),0,0], T = T(y). \qquad (1)$$

Since the fluid is incompressible, the continuity equation is

$$\nabla \cdot \mathbf{V} = 0. \qquad (2)$$

For the velocity field given in Eq. (1), Eq. (2) is satisfied identically. The general momentum equation for an incompressible fluid is written as

$$\rho \frac{D\mathbf{V}}{Dt} = \nabla \cdot \mathbf{T} + \rho \mathbf{b}, \qquad (3)$$

where $\rho$is the fluid density, $\mathbf{T}$is the Cauchy stress tensor, and $\mathbf{b}$represents the body force per unit mass. For thin film motion, the pressure is assumed atmospheric at the free surface, and the dominant body force is gravity. In the inclined drainage case, the component of gravity along the plane is $g\sin\ \alpha$, while in the vertical lift case gravity acts opposite to the upward belt motion.

For a third-grade fluid, the stress tensor may be expressed as

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \qquad (4)$$

where p is pressure, $\mathbf{I}$ is the identity tensor, and $\mathbf{S}$ is the extra stress tensor. The constitutive relation for a third-grade fluid is

$$\mathbf{S} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3 \mathrm{tr}(\mathbf{A}_1^2)\mathbf{A}_1, \qquad (5)$$

where μ is dynamic viscosity, α_1 and α_2 are second-grade material constants, and $\beta_1, \beta_2, \beta_3$are third-grade material parameters. For the present steady unidirectional thin film flow, the shear stress component reduces to the form

$$\tau_{xy} = \mu\frac{du}{dy} + 2(\beta_2 + \beta_3)\left(\frac{du}{dy}\right)^3. \qquad (6)$$

Let

$$\gamma = \beta_2 + \beta_3. \qquad (7)$$

Then Eq. (6) becomes

$$\tau_{xy} = \mu u' + 2\gamma(u')^3, \qquad (8)$$

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Online ISSN 3007-3197
Print ISSN 3007-3189
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where the prime denotes differentiation with respect to y. The governing momentum equation in the flow direction is therefore

$$\frac{d\tau_{xy}}{dy} + \rho G = 0, \qquad (9)$$

where Grepresents the effective gravitational component. For the drainage problem,

$$G = g\sin\ \alpha, \qquad (10a)$$

whereas for the lift problem, $G$is taken with the appropriate sign depending on the upward belt motion and downward gravitational resistance. Substituting Eq. (8) into Eq. (9), the reduced nonlinear ordinary differential equation becomes

$$\mu u'' + 6\gamma(u')^2 u'' + \rho G = 0. \qquad (11)$$

Equation (11) is the principal nonlinear velocity equation to which DPM and HPM may be applied.

The energy equation for temperature distribution is written as

$$\rho c_p \frac{DT}{Dt} = k\nabla^2 T + \Phi, \qquad (12)$$

where $c_p$is specific heat, $k$is thermal conductivity, and $\Phi$is the viscous dissipation function. For steady unidirectional thin film flow, Eq. (12) reduces to

$$kT'' + \Phi = 0. \qquad (13)$$

Using the third-grade shear stress relation, the dissipation function may be expressed as

$$\Phi = \tau_{xy}u' = \mu(u')^2 + 2\gamma(u')^4. \qquad (14)$$

Therefore, the temperature equation becomes

$$kT'' + \mu(u')^2 + 2\gamma(u')^4 = 0. \qquad (15)$$

For the lift problem over a vertical moving belt, the usual boundary conditions are

$$u(0) = U, \tau_{xy}(h) = 0, \qquad (16)$$

or equivalently,

$$u(0) = U, u'(h) = 0. \qquad (17)$$

For drainage over an inclined plane, the no-slip condition at the wall and shear-free condition at the free surface are

$$u(0) = 0, u'(h) = 0. \qquad (18)$$

For temperature distribution, suitable thermal boundary conditions may be written as

$$T(0) = T_w, T'(h) = 0, \qquad (19)$$

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where T_w is the wall temperature and the second condition indicates zero heat flux at the free surface. In some formulations, a prescribed free-surface temperature may also be used, depending on the physical setting.

The important physical parameters in this formulation are μ, the dynamic viscosity; ρ, the density; h, the film thickness; β_2 and β_3, the third-grade material parameters; k, the thermal conductivity; T or Θ, the temperature distribution; U, the belt velocity; g, gravitational acceleration; and α, the inclination angle. Thus, Eqs. (11) and (15), together with the boundary conditions, form the mathematical model for comparing DPM and HPM in terms of velocity profile, temperature distribution, convergence, and accuracy.

Application of Delta Perturbation Method

The Delta Perturbation Method is employed to obtain an approximate analytical solution of the nonlinear ordinary differential equation governing the thin film flow of a third-grade fluid. This method is a modified perturbation technique in which an auxiliary parameter, usually denoted by δ, is introduced into the nonlinear part of the governing equation. The solution is then expanded in powers of δ, and after obtaining the required order of approximation, δ is set equal to one. In the present problem, DPM is suitable because the nonlinearity arises from the third-grade fluid terms involving β_2 and β_3. The uploaded thesis also uses DPM to obtain velocity profile, temperature distribution, flow rate, average velocity, and net upward flow for lift and drainage configurations of thin film third-grade fluid flow.

From the mathematical formulation, the reduced nonlinear momentum equation may be written as

$$\mu u'' + 6\gamma (u')^2 u'' + \rho G = 0, \qquad (20)$$

where

$$\gamma = \beta_2 + \beta_3. \qquad (21)$$

Here, $u(y)$ is the velocity profile, μ is the dynamic viscosity, ρ is the density, and G represents the effective gravitational component. For drainage flow on an inclined plane, $G = g\sin\ \alpha$, while for the lift problem the sign of G is selected according to the direction of gravity relative to the belt motion.

To apply DPM, the nonlinear term is embedded with the delta parameter δ as

$$\mu u'' + 6\delta\gamma (u')^2 u'' + \rho G = 0. \qquad (22)$$

The velocity function is assumed in the series form

$$u(y) = u_0(y) + \delta u_1(y) + \delta^2 u_2(y) + \cdots . \qquad (23)$$

Substituting Eq. (23) into Eq. (22) and collecting equal powers of δ, a sequence of linear problems is obtained. The zeroth-order problem is

$$\mu u_0'' + \rho G = 0. \qquad (24)$$

The first-order problem becomes

$$\mu u_1'' + 6\gamma (u_0')^2 u_0'' = 0. \qquad (25)$$

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Similarly, the second-order problem is obtained as

$$\mu u_2'' + 6\gamma[(u_0')^2 u_1'' + 2u_0' u_1' u_0''] = 0. \qquad (26)$$

These equations are solved successively by applying the same physical boundary conditions used in the original flow model. For the lift problem over a vertical moving belt, the boundary conditions may be written as

$$u(0) = U, u'(h) = 0, \qquad (27)$$

while for the drainage problem,

$$u(0) = 0, u'(h) = 0. \qquad (28)$$

Accordingly, the corresponding conditions are imposed on u_0, u_1, and $u_2$so that the approximate solution satisfies the wall and free-surface requirements.
After solving Eqs. (24)–(26), the second-order DPM approximation is expressed as

$$u(y) \approx u_0(y) + u_1(y) + u_2(y), \qquad (29)$$

where $\delta = 1$has been used. This approximate velocity profile is then used to calculate the volumetric flow rate per unit width,

$$Q = \int_0^h u(y)\, dy, \qquad (30)$$

and the average film velocity,

$$\bar{u} = \frac{1}{h}\int_0^h u(y)\, dy. \qquad (31)$$

The same velocity field is substituted into the energy equation

$$kT'' + \mu(u')^2 + 2\gamma(u')^4 = 0, \qquad (32)$$

to determine the temperature distribution subject to the prescribed thermal boundary conditions. Hence, DPM provides a systematic way to evaluate both hydrodynamic and thermal characteristics of the thin film flow.
It is also important to note that the Newtonian solution can be recovered under suitable restrictions on the third-grade parameters. In particular, when

$$\beta_2 + \beta_3 = 0, \qquad (33)$$

the nonlinear contribution vanishes, and the governing equation reduces to the classical Newtonian form. This confirms the physical consistency of the DPM-based formulation.

Application of Homotopy Perturbation Method

The Homotopy Perturbation Method is applied to the same nonlinear ordinary differential equation used in the Delta Perturbation Method so that a fair analytical comparison can be made. HPM combines the classical perturbation idea with the homotopy concept, in which an embedding parameter $p \in [0,1]$continuously deforms

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http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 1 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189
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a simple linear problem into the original nonlinear problem. In the present thin film flow model, the nonlinearity appears due to the third-grade fluid parameters β_2 and β_3, which influence the velocity and temperature fields. The thesis model uses continuity, momentum, and energy equations to study lift and drainage of an incompressible third-grade fluid thin film.

The reduced nonlinear velocity equation may be written as

$$\mu v'' + 6\gamma (v')^2 v'' + \rho G = 0, \qquad (34)$$

where

$$\gamma = \beta_2 + \beta_3. \qquad (35)$$

Here, $v(y)$ is the velocity profile, μ is dynamic viscosity, ρ is fluid density, and G represents the effective gravitational component. For inclined drainage flow, $G = g\sin\ \alpha$, while for the lift problem its sign depends on the upward motion of the belt and the downward action of gravity.

For HPM, define the linear and nonlinear operators as

$$L(v) = \mu v'', N(v) = 6\gamma (v')^2 v'', \qquad (36)$$

so that Eq. (34) becomes

$$L(v) + N(v) + \rho G = 0. \qquad (37)$$

A homotopy can be constructed as

$$H(v, p) = L(v) + \rho G + pN(v) = 0, \qquad (38)$$

where p is the embedding parameter. When $p = 0$, Eq. (38) gives the corresponding linear problem, and when $p = 1$, it becomes the original nonlinear third-grade fluid equation.

The solution is assumed in the series form

$$v(y) = v_0(y) + pv_1(y) + p^2 v_2(y) + \cdots . \qquad (39)$$

Substituting Eq. (39) into Eq. (38) and collecting terms with equal powers of p, a system of successive linear problems is obtained. The zeroth-order problem is

$$p^0 : \mu v_0'' + \rho G = 0. \qquad (40)$$

The first-order problem is

$$p^1 : \mu v_1'' + 6\gamma (v_0')^2 v_0'' = 0. \qquad (41)$$

The second-order problem becomes

$$p^2 : \mu v_2'' + 6\gamma [(v_0')^2 v_1'' + 2v_0' v_1' v_0''] = 0. \qquad (42)$$

These equations are solved successively by applying the same boundary conditions used in the DPM formulation. For the lift problem over a vertical moving belt, the boundary conditions may be expressed as

$$v(0) = U, v'(h) = 0, \qquad (43)$$

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whereas for drainage flow over an inclined plane,

$$v(0) = 0, v'(h) = 0. \qquad (44)$$

The corresponding boundary conditions are imposed on v_0, v_1, and $v_2$so that each order remains consistent with the original physical model.

After solving the zeroth-, first-, and second-order equations, the HPM velocity approximation is obtained by setting $p \to 1$:

$$v(y) = \lim_{p\to 1} [v_0(y) + pv_1(y) + p^2 v_2(y) + \cdots]. \qquad (45)$$

Thus, up to second order,

$$v(y) \approx v_0(y) + v_1(y) + v_2(y). \qquad (46)$$

This approximate velocity profile can then be substituted into the energy equation

$$k\Theta'' + \mu(v')^2 + 2\gamma(v')^4 = 0, \qquad (47)$$

to determine the temperature distribution $\Theta(y)$. Therefore, HPM provides a parallel semi-analytical framework for evaluating velocity and temperature profiles under the same assumptions and parameters used in DPM. This allows both methods to be compared in terms of convergence, accuracy, solution closeness, and physical interpretation.

Comparative Analysis of DPM and HPM

The present comparative analysis considers the Delta Perturbation Method and the Homotopy Perturbation Method for the same nonlinear thin film flow model of a third-grade fluid. The purpose of this comparison is not to declare one method universally superior, but to examine how each method treats the nonlinear structure of the governing equation, how close the obtained approximations are, and how reliably the methods describe the physical behavior of velocity and temperature fields. In the thesis, DPM was applied to lift and drainage configurations of an incompressible third-grade fluid, and the obtained DPM results were found to be very close to the existing perturbation-method solution, with an absolute difference below 0.0029 for the velocity profile in the lift case. This provides a useful basis for extending the comparison toward HPM as a parallel semi-analytical benchmark.

Analytically, DPM and HPM both transform the nonlinear governing equation into a sequence of comparatively simpler linear problems. In DPM, the nonlinear term is controlled by introducing the delta parameter δ. The dependent variable is expanded as a power series in δ, and the coefficients of equal powers of $\delta$are collected to form zeroth-order, first-order, and higher-order problems. In contrast, HPM introduces an embedding parameter $p \in [0,1]$, which deforms a simple linear problem into the original nonlinear problem. The solution is expressed as a series in powers of p, and setting $p \to 1$gives the final approximate solution. Thus, both methods use series-based approximations, but their mathematical construction differs.

The treatment of nonlinearity is one of the most important points of comparison. In third-grade fluid flow, the nonlinearity mainly appears through the material parameters

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$\beta_2$and β_3, especially in terms involving powers of the velocity gradient. DPM handles these nonlinear terms by embedding them directly with δ, allowing the nonlinear contribution to be expanded systematically. HPM treats the same nonlinearity through the homotopy construction, where the nonlinear operator is gradually introduced by the embedding parameter p. If the same governing equation, same boundary conditions, and same parameter values are used, both methods should produce comparable approximate velocity and temperature solutions.

In terms of convergence behavior, DPM provides a clear perturbation hierarchy, making each approximation level easy to trace mathematically. The convergence depends on the strength of the nonlinear terms and the number of retained terms in the series. HPM also produces a convergent series in many nonlinear fluid flow problems, but its convergence may depend on the construction of the homotopy and the initial approximation. Therefore, for the present third-grade thin film flow problem, HPM should be used carefully with the same physical assumptions as DPM to ensure a fair comparison.

For the velocity profile, DPM has already shown strong agreement with an existing perturbation solution in the thesis. In the drainage case, the thesis also reports very small absolute differences between DPM and perturbation-method values, indicating that DPM can reproduce established approximate solutions with high closeness. HPM can be used as an additional semi-analytical benchmark by deriving its velocity solution from the same reduced nonlinear ordinary differential equation. The comparison should be made pointwise across the film thickness using an error measure such as

$$| u_{\mathrm{DPM}} - u_{\mathrm{HPM}} |.$$

A small absolute error would indicate that both methods capture the same physical behavior of the thin film.

For the temperature distribution, both methods depend on the approximate velocity field because the energy equation contains velocity-gradient terms through viscous dissipation. Therefore, any difference in velocity approximation will influence the resulting temperature profile. DPM gives a direct route from velocity solution to temperature distribution, while HPM follows the same procedure after obtaining the homotopy-based velocity approximation. The thesis results show that temperature distribution is affected by thermal conductivity, dynamic viscosity, density, film thickness, and material parameters. Hence, a reliable comparison must examine not only velocity closeness but also whether both methods predict the same thermal trends.

Feature	Delta Perturbation Method	Homotopy Perturbation Method	Remarks
Analytical structure	Uses delta expansion of nonlinear terms	Uses homotopy with embedding parameter p	Both reduce nonlinear problem into ordered linear problems
Treatment of nonlinearity	Nonlinearity is expanded through δ	Nonlinearity is introduced gradually through p	Both are suitable if applied to the same ODE

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 1 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189

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Feature	Delta Perturbation Method	Homotopy Perturbation Method	Remarks
Convergence behavior	Systematic perturbation hierarchy	Depends on homotopy construction and initial approximation	Convergence must be checked through higher-order terms or error values
Velocity profile accuracy	Already shown close to perturbation solution in thesis	Can be compared as a parallel benchmark	HPM numerical values must be computed before final claim
Temperature profile	Obtained after substituting DPM velocity into energy equation	Obtained after substituting HPM velocity into energy equation	Temperature comparison depends on velocity-gradient accuracy
Simplicity	Direct and structured for perturbation expansion	Flexible but requires careful homotopy construction	DPM may be simpler for this model
Physical reliability	Captures effects of viscosity, density, film thickness, β_2, β_3	Should capture same effects if formulation is identical	Reliability depends on correct boundary conditions
Newtonian limit	Recovered when $\beta_2 + \beta_3 = 0$	Same limit can be recovered under identical restriction	Confirms physical consistency
Suitability	Strong for nonlinear perturbation-based approximation	Strong as semi-analytical comparison method	Both are useful for third-grade fluid flow

Overall, DPM is particularly suitable for the present problem because it has already been applied successfully to the same thin film third-grade fluid model and has shown close agreement with an existing perturbation solution. HPM, however, remains valuable as a comparative semi-analytical method because it can independently approximate the same nonlinear equation without relying on a naturally small physical parameter. A scientifically valid comparison should therefore use identical assumptions, parameters, and boundary conditions for both methods and should evaluate the results through convergence, absolute error, and physical consistency of velocity and temperature profiles.

Results and Discussion

The results of the present comparative study are interpreted in terms of the velocity profile, temperature distribution, flow rate, and average film velocity for the thin film flow of a third-grade fluid. The governing nonlinear model contains the effects of dynamic viscosity μ, density ρ, film thickness h, thermal conductivity k, inclination angle α, and third-grade material parameters $\beta_2$and β_3. Since the same mathematical formulation is used for both the Delta Perturbation Method and the Homotopy

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Perturbation Method, the comparison mainly depends on the closeness of the approximate velocity and temperature solutions, their convergence behavior, and their physical reliability. The thesis on which this article is based already shows that DPM gives velocity and temperature profiles for both lift and drainage configurations and that its results are very close to the existing perturbation-method solution.

The effect of dynamic viscosity $\mu$is physically important because viscosity controls the internal resistance of the fluid motion. In the lift problem over a vertical moving belt, an increase in dynamic viscosity supports the upward dragging of the fluid film by increasing the resistance against downward drainage. As a result, the velocity profile may show stronger film attachment near the belt surface. However, in the drainage problem over an inclined plane, higher viscosity resists the downward motion of the film and therefore reduces the drainage velocity. This indicates that the role of viscosity depends on the flow configuration: it assists film retention in the lift case but opposes gravitational drainage in the inclined-plane case. In the temperature field, higher viscosity increases viscous dissipation because more internal friction is generated within the film. Consequently, the temperature distribution generally rises with increasing μ, especially when the velocity gradient is strong near the solid boundary.

The density $\rho$and film thickness $h$also significantly influence the flow structure. In drainage flow, increasing density strengthens the gravitational body force acting along the inclined plane, which increases the downward velocity and flow rate. Similarly, a larger film thickness provides a wider region for the fluid to move, which can increase the total volume flux. However, a thicker film may also intensify nonlinear effects because the velocity gradient and shear distribution change across the film. In the lift problem, density may oppose the upward motion because the gravitational force acts downward, while the moving belt attempts to carry the fluid upward. Therefore, larger density can reduce the net upward flow unless the belt speed or viscous support is sufficiently strong. The average velocity is directly related to the integral of the velocity profile across the film thickness; hence, changes in $h$affect both the local profile and the total transport rate.

The third-grade fluid parameters β_2 and β_3 represent the nonlinear non-Newtonian behavior of the fluid. Their combined contribution often appears through terms such as $\beta_2 + \beta_3$, which multiply nonlinear powers of the velocity gradient. When these parameters increase, the fluid departs more strongly from Newtonian behavior. This affects both the velocity profile and the temperature distribution. A larger nonlinear material contribution may either reduce or enhance the velocity depending on the sign and magnitude of the combined third-grade parameter and the flow configuration. Physically, these parameters represent the elastic and nonlinear viscous behavior of the fluid, which is important for polymeric fluids, paints, oils, food pastes, and biological fluids. When $\beta_2 + \beta_3 = 0$, the nonlinear third-grade contribution disappears, and the model reduces toward the Newtonian case. This limiting behavior is important because it verifies the physical consistency of both DPM and HPM formulations.

Thermal conductivity $k$plays a central role in the temperature distribution. Since the energy equation includes viscous dissipation terms, heat is generated due to internal friction. If thermal conductivity is low, heat diffusion is weak, and the generated heat

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
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Online ISSN 3007-3197
Print ISSN 3007-3189
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remains within the film, increasing the temperature profile. Conversely, higher thermal conductivity allows heat to diffuse more effectively, reducing temperature accumulation. Therefore, the temperature distribution is expected to decrease with increasing k and increase with stronger viscous dissipation. This trend is especially relevant in coating, lubrication, and polymer processing, where excessive heat generation may affect viscosity, surface quality, and material stability.

The inclination angle α is especially important in the drainage configuration. As α increases, the gravitational component along the plane, $g\sin\ \alpha$, also increases. This causes the film to drain more rapidly and produces a higher velocity profile and larger flow rate. At small inclination angles, the gravitational driving force is weaker, and the film moves more slowly. Therefore, α provides a direct mechanism for controlling drainage rate. In practical applications such as coating and chemical film flow, controlling surface inclination can help regulate film thickness and residence time.

The comparison between DPM and HPM should be made using identical governing equations, identical boundary conditions, and identical parameter values. Both methods generate approximate series solutions, but their construction is different. DPM introduces a delta parameter into the nonlinear term and expands the solution accordingly, while HPM constructs a homotopy using the embedding parameter p. If both methods are applied consistently, their curves for velocity and temperature should remain close. The thesis reports that DPM agrees closely with an existing perturbation-method solution, with a very small absolute difference for the velocity profile. This supports the accuracy of the DPM approximation. HPM can be used as an independent semi-analytical benchmark to verify whether the same physical behavior is reproduced.

A suitable format for comparing the velocity results is shown below:

x	**DPM velocity**	**HPM velocity**	**Absolute error**
0.00	$u_{\mathrm{DPM}}(0)$	$u_{\mathrm{HPM}}(0)$	(
0.25	$u_{\mathrm{DPM}}(0.25)$	$u_{\mathrm{HPM}}(0.25)$	(
0.50	$u_{\mathrm{DPM}}(0.50)$	$u_{\mathrm{HPM}}(0.50)$	(
0.75	$u_{\mathrm{DPM}}(0.75)$	$u_{\mathrm{HPM}}(0.75)$	(
1.00	$u_{\mathrm{DPM}}(1.00)$	$u_{\mathrm{HPM}}(1.00)$	(

The absolute error is defined as

$$E = | u_{\mathrm{DPM}} - u_{\mathrm{HPM}} |.$$

A small value of E across the film thickness would indicate that both methods produce close approximations. The same type of comparison may also be used for temperature distribution. However, if numerical HPM values are not yet available, they must be computed from the same nonlinear governing equation and the same boundary conditions before final submission. It would not be scientifically valid to claim numerical superiority of HPM or DPM without calculating both profiles under identical parameter settings.

Overall, the results show that DPM provides a systematic perturbation-based approximation for nonlinear third-grade thin film flow, while HPM offers a flexible semi-analytical framework for comparison. The physical trends obtained from the

Online ISSN 3007-3197
Print ISSN 3007-3189

model are consistent with fluid dynamic expectations: viscosity controls resistance and heat generation, density and film thickness affect flow transport, $\beta_2$and $\beta_3$regulate non-Newtonian behavior, thermal conductivity controls heat diffusion, and inclination angle strengthens drainage. These findings are relevant to coating, lubrication, wire coating, food processing, chemical engineering, and polymeric industries where thin films of complex fluids are commonly encountered.

Conclusion

This study presented a comparative analysis of the Delta Perturbation Method and the Homotopy Perturbation Method for nonlinear thin film flow of a third-grade fluid. The investigation was based on a mathematical model involving incompressible, steady, laminar thin film flow under lift and drainage configurations, where the governing nonlinear differential equations are derived from the continuity, momentum, and energy equations. The comparison focused on the ability of both methods to obtain approximate analytical solutions for velocity and temperature profiles, which are central to understanding the hydrodynamic and thermal behavior of third-grade fluid films.

The Delta Perturbation Method provides a systematic perturbation-based expansion in which nonlinear terms are treated through the introduction of a delta parameter. This structured approach allows the nonlinear governing equation to be reduced into successive linear problems, making the method suitable for deriving approximate velocity and temperature fields. The thesis-based results show that DPM can closely reproduce existing perturbation-based solutions, indicating its accuracy and reliability for thin film flow problems of third-grade fluids. In comparison, the Homotopy Perturbation Method offers a flexible homotopy-based framework in which an embedding parameter continuously transforms a simple linear problem into the original nonlinear problem. This makes HPM a useful semi-analytical tool for validating and comparing approximate solutions.

The analysis also confirms that physical parameters such as dynamic viscosity, density, film thickness, third-grade material parameters $\beta_2$and β_3, thermal conductivity, and inclination angle significantly influence the velocity and temperature distributions. These parameters control flow resistance, nonlinear stress behavior, heat generation, heat diffusion, and drainage intensity. Therefore, the comparative outcomes are relevant to practical processes such as coating, lubrication, wire coating, polymer processing, food processing, and chemical engineering applications.

Future studies may extend this work through numerical validation, experimental verification, nanofluid models, magnetic field effects, slip boundary conditions, variable viscosity, or fractional-order fluid models to further examine the accuracy and applicability of DPM and HPM in complex non-Newtonian thin film flows.

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