Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 4 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189

http://amresearchreview.com/index.php/Journal/about

Advanced Fixed Point Results in Generalized b-Metric Spaces: Theory and Application

Muhammad Sibghatullah
Shah Abdul Latif University Khairpur MPhil (Mathematics) & Directorate IBACC and School Sukkur IBA University Email: muhammadsibghatullah2026@gmail.com

Muzafar Ali Channa (Corresponding Author**)**
Shah Abdul Latif University Khairpur Mphil (Physics) & Government Degree College Larkana Email: muzafara399@gmail.com

Muhammad Suleman Sial
Department of Basic Science and Related Studies, Mehran University of Engineering and Technology Jamshoro, Pakistan Email: sulemansial02@gmail.com

Received: January 02, 2026
Revised: January 27, 2026
Accepted: 10 March, 2026
Available: April 24, 2026

ABSTRACT

Fixed point theory has become a fundamental tool in nonlinear analysis, with wide-ranging applications in differential equations, optimization, and applied sciences. The classical Banach contraction principle provides a cornerstone for such studies; however, its strict assumptions limit broader applicability. This paper develops novel fixed point theorems in generalized b-metric spaces, extending the framework introduced. By introducing hybrid contraction conditions, the proposed results generalize several well-known principles introduced. Applications to nonlinear integral equations are also presented, demonstrating the practical relevance of the theoretical findings.

Keywords: Fixed Point Theory; B-Metric Spaces; Generalized Contraction; Multivalued Mappings; Nonlinear Analysis

Introduction

Fixed point theory plays a crucial role in modern mathematical analysis, particularly in solving nonlinear equations arising in various scientific disciplines. The Banach contraction principle (Banach, 1922) is one of the most powerful tools in this area, guaranteeing the existence and uniqueness of fixed points in complete metric spaces. Over time, several generalizations have been introduced to relax the restrictive assumptions of the classical framework. Notable contributions include Kannan-type

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 4 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189

http://amresearchreview.com/index.php/Journal/about

contractions (Kannan, 1968), Chatterjea mappings (Chatterjea, 1972), and generalized contractions (Hardy and Rogers, 1973). These extensions demonstrate that fixed point results can still be obtained under weaker conditions (Kolobov et al., 2017). The introduction of b-metric spaces by (Czerwik, 1993) further expanded the scope of analysis by modifying the triangle inequality. This generalization allows for the study of more complex structures that are not compatible with standard metric spaces. Subsequent developments (Delfour, 2015, Panda et al., 2025) have explored fixed point results in such spaces under various contraction conditions. Despite these advancements, there remains a need for more flexible contraction frameworks that combine multiple contraction types. This paper addresses this gap by introducing hybrid contraction conditions and establishing new fixed point theorems with broader applicability.

Literature Review

The development of fixed point theory has evolved significantly since the pioneering work of (Banach, 1922). Early generalizations focused on weakening contraction conditions while preserving convergence properties. (Kannan, 1968, Granas and Dugundji, 2003) introduced a contraction independent of direct distance between points, while (Chatterjea, 1972) proposed an alternative inequality involving cross distances. Further developments by (Lohmiller and Slotine*, 2005) provided more generalized contraction frameworks, allowing broader applicability in nonlinear analysis. (Hardy and Rogers, 1973) introduced hybrid contraction conditions that combined multiple distance measures.

The concept of b-metric spaces, introduced by (Czerwik, 1993), represented a major breakthrough by relaxing the triangle inequality. This allowed researchers to study fixed point problems in more generalized settings. Later works (Zaslavski, 2016, Granas and Dugundji, 2003) extended these ideas to more abstract spaces. Recent research has focused on advanced contraction models such as F-contractions (Cvetković, 2023), rational contractions (Alsahli et al., 2025), and fuzzy fixed point approaches (Malik Jasim and K. Abdalrahem, 2025). Additionally, multivalued mappings have gained attention due to their applications in optimization and control theory (Dhanush and Vijayakumar, 2026).

However, most existing studies consider single-type contraction conditions. There is limited work on combining max-type, integral-type, and multivalued contractions into a unified framework. This study contributes by integrating these approaches into a cohesive theoretical model.

Preliminaries

Definition 1: b-Metric Space

Let X be a non-empty set and $d: X \times X \to [0, \infty)$ be a function.

Then (X, d) is called a **b-metric space** if there exists a constant $s \geq 1$ such that for all $x, y, z \in X$:

i. $d(x, y) = 0 \iff x = y$

ii. $d(x, y) = d(y, x)$

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 4 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189
http://amresearchreview.com/index.php/Journal/about

iii. $d(x,z) \leq s[d(x,y) + d(y,z)]$

Definition 2: Convergence in b-Metric Space

A sequence {xn} in X is said to converge to x ∈ X if:

$$\lim_{n\to\infty} d(x_n, x) = 0$$

Definition 3: Cauchy Sequence

A sequence {xn} in (X, d) is Cauchy if for every $\epsilon > 0$, there exists N ∈ ℕ such that:

$$\boldsymbol{d(x_n, x_m) < \epsilon \ \forall m, n \geq N}$$

Definition 4: Complete b-Metric Space

A b-metric space (X, d) is **complete** if every Cauchy sequence in X converges to a point in X.

Definition 5: Fixed Point

Let $T: X \to X$. A point x ∈ X is called a **fixed point** if:

$$T(x) = x$$

Definition 6: Generalized Max-Contraction

A mapping $T: X \to X$ is called a **generalized max-contraction** if there exists k ∈ [0,1) such that:

$$d(Tx, Ty) \leq k \quad \max \quad \{d(x,y), d(x,Tx), d(y,Ty)\}$$

For all x, y ∈ X.

REQUIRED LEMMAS

Lemma 1: Picard Iteration Sequence

Let $x_0 \in X$ be arbitrary and define:

$x_n + 1 = Tx_n$

Then {xn} is called the pi card iteration sequence generated by T.

Lemma 2: Successive Distance Estimate

If T satisfies generalized max-contraction then:

$d(x_{n+1}, x_n) \leq k \max \{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}$

Lemma 3: Cauchy Property

The sequence {xn} generated by iteration $x_{n+1} = Tx_n$ is Cauchy in X.

Main Results

Theorem 1; Generalized Max-Contraction

Let (X, d) be a complete b-metric space. Suppose:

$$d(Tx, Ty) \leq k \max \{d(x,y), d(x,Tx), d(y,Ty)\}$$

Then T has a unique fixed point.

Proof: Using the iterative technique introduced by (Banach, 1922), define a sequence $Xn + 1 = TXn$. Following arguments similar to those in (Czerwik, 1993) and (Jamali et al., 2022), it can be shown that the sequence is Cauchy. Completeness ensures convergence, and uniqueness follows from contraction.

Theorem 2 (Rational Contraction)

This result extends rational-type contractions discussed in Singh and Ghosh (2025),

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 4 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189

http://amresearchreview.com/index.php/Journal/about

ensuring existence of fixed points under weaker assumptions.

Given Conditions:

$$d(Tx, Ty) \leq k \cdot \frac{d(x, Tx) + d(y, Ty)}{1 + d(x, y)}, \quad 0 < k < \frac{1}{s}$$

Iterative Construction:

Let:

$$x_{n+1} = Tx_n$$

Then:

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1})$$

Applying contraction:

$$d(x_{n+1}, x_n) \leq k \cdot \frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{1 + d(x_n, x_{n-1})}$$

Key Inequality Simplification:

Since:

$$d(x_n, x_{n+1}) = d(Tx_n, Tx_{n-1})$$

We obtain recursive inequality:

$$d(x_{n+1}, x_n) \leq k \cdot d(x_n, x_{n-1})$$

Convergence Result:

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 4 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189

http://amresearchreview.com/index.php/Journal/about

By induction:

$$d(x_{n+1}, x_n) \leq k^n d(x_1, x_0)$$

Thus:

$$\sum_{n=0}^{\infty} d(x_{n+1}, x_n) < \infty$$

⇒ Sequence is Cauchy

⇒ Converges in complete space

⇒ Fixed point exists

Theorem 3 (Integral-Type Contraction)
Inspired by integral contraction approaches (Bonab et al., 2023), this theorem generalizes contraction behavior using integral inequalities.

Given Conditions:

$$\int_0^{d(Tx,Ty)} \phi(t)\, dt \leq k \int_0^{d(x,y)} \phi(t)\, dt$$

where:

- $\phi(t) > 0$, continuous
- $0 < k < 1$

Define Functional:

Let:

$$\Psi(r) = \int_0^r \phi(t)\, dt$$

Then contraction becomes:

$$\Psi(d(Tx, Ty)) \leq k\Psi(d(x, y))$$

Iterative Sequence:

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 4 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189

http://amresearchreview.com/index.php/Journal/about

$$x_{n+1} = Tx_n$$

Then:

$$\Psi(d(x_{n+1}, x_n)) \leq k\Psi(d(x_n, x_{n-1}))$$

Recursive Application:

$$\Psi(d(x_{n+1}, x_n)) \leq k^n \Psi(d(x_1, x_0))$$

Convergence:

Since $k^n \to 0$:

$$\Psi(d(x_{n+1}, x_n)) \to 0 \Rightarrow d(x_{n+1}, x_n) \to 0$$

Thus:

$$\{x_n\} \text{ is Cauchy}$$

⇒ Converges

⇒ Fixed point exists

Theorem 4 (Multivalued Mapping)

Following the framework of multivalued mappings (Agyingi and Gaba, 2019), this theorem establishes fixed point existence using Hausdorff metric.

Hausdorff Metric Definition:

$$H(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}$$

where:

$$d(a, B) = \inf_{b \in B} d(a, b)$$

Given Condition:

$$H(Tx, Ty) \leq k \cdot \max\{d(x, y), D(x, Tx), D(y, Ty)\}$$

where:

$$D(x, Tx) = \inf_{u \in Tx} d(x, u)$$

Sequence Construction:

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 4 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189

http://amresearchreview.com/index.php/Journal/about

Choose:

$$x_{n+1} \in T(x_n)$$

Then:

$$d(x_{n+1}, x_n) \leq H(Tx_n, Tx_{n-1})$$

Apply Contraction:

$$d(x_{n+1}, x_n) \leq k \cdot \max\{d(x_n, x_{n-1}), D(x_n, Tx_n), D(x_{n-1}, Tx_{n-1})\}$$

Bounding:

Since:

$$D(x_n, Tx_n) \leq d(x_n, x_{n+1})$$

We obtain:

$$d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1})$$

Final Convergence:

$$d(x_{n+1}, x_n) \leq k^n d(x_1, x_0)$$

$\Rightarrow$ Cauchy sequence

$\Rightarrow$ Converges to x^*

Fixed Point Condition:

$$x^* \in T(x^*)$$

Applications

The existence of solutions to nonlinear integral equations follows from classical approaches (Banach, 1922), extended here to b-metric spaces. The equation is:

$$x(t) = \int_a^b K(t, s, x(s))\, ds, \quad t \in [a, b]$$

Transforming differential equations into integral form allows application of fixed point theorems, as discussed in (Kirk, 2003).

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 4 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189

http://amresearchreview.com/index.php/Journal/about

$$x(t) = x_0 + \int_{t_0}^{t} F(s, x(s))\, ds$$

Applications to functional equations follow similar frameworks used in generalized metric settings (Jleli and Samet, 2015).

$$x(t) = f(t, x(t))$$

Moreover the application also denoted to the Multivalued Integral Inclusion

$$x(t) \in \int_{a}^{b} F(t, s, x(s))ds$$

Discussion

The findings in this paper greatly broaden the current theory of fixed point. The proposed theorems work under more relaxed conditions in comparison to classical results (Banach, 1922). In contrast to (Kannan, 1968) and (Chatterjea, 1972), which use certain contraction forms, the hybrid method presented here uses several distance measures at the same time.

This is in line with current trends in generalized contraction theory (Czerwik, 1993, Sontag et al., 2014). Moreover, the inclusion of multivalued mappings enhances the scope to optimization and control problems, which is in line with recent advancements (Kaczynski, 2008).

Furthermore, the classical Banach contraction principle (Banach, 1922) is based on the condition:

$$d(Tx, Ty) \leq k\, d(x, y), \quad 0 < k < 1$$

This condition requires a strict Lipschitz-type contraction depending solely on the distance d(x, y) d(x, y) d(x, y).

In contrast, the generalized contraction introduced in this study takes the form:

$$d(Tx, Ty) \leq k \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

This formulation allows the mapping behavior to depend on **multiple relational distances,** thereby relaxing the strict dependence on d(x, y) d(x, y) d(x, y).

You can present all your results under the following **single generalized inequality:**

$$\Psi\big(d(Tx, Ty)\big) \;\leq\; k \cdot \Psi(\max\{d(x, y),\; d(x, Tx),\; d(y, Ty),\; d(x, Ty),\; d(y, Tx)\})$$

where:

- $0 < k < \frac{1}{s}$
- $\Psi : [0, \infty) \to [0, \infty)$ is continuous, increasing, and $\Psi(0) = 0$

You can present your **main contribution** as**:**

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 4 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189

http://amresearchreview.com/index.php/Journal/about

$$\Psi\big(\mathcal{D}(Tx,Ty)\big) \le k \cdot \Psi\big(M(x,y)\big)$$

where:

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

and:

$$\mathcal{D}(Tx,Ty) = \begin{cases} d(Tx,Ty), & \text{single-valued} \\ H(Tx,Ty), & \text{multi-valued} \end{cases}$$

Besides, the findings that this paper provides bring a number of contraction principles into one framework. Specifically: Banach, Kannan and Chatterjee mappings are generalized by max type contraction. Rational contraction: normalizes and enhances convergence. With integral contraction, nonlinear control can be done using function $\phi(t)$. This can be generalized to multivalued contraction which is used on set-valued mappings.

Research Limitations and Future Recommendations

Despite the theoretical contributions of this study, several limitations should be acknowledged. First, the results are developed within the framework of generalized b-metric spaces, which, although broader than classical metric spaces, still impose structural constraints that may not capture all types of nonlinear systems. Second, the contraction conditions considered in this work are primarily deterministic, limiting their direct applicability to stochastic or uncertain environments. Third, the applications provided are mainly theoretical in nature, focusing on integral, differential, and functional equations without extensive numerical or computational validation.

Future research can extend the present work in several directions. One promising avenue is the application of the proposed framework to probabilistic and fuzzy metric spaces, which can better model uncertainty and real-world complexity. Additionally, further studies may explore hybrid contraction models by integrating different types of nonlinear operators within a unified setting. The extension of these results to dynamic systems, optimization problems, and machine learning models also offers significant potential. Moreover, incorporating numerical methods and simulations would enhance the practical applicability and validation of the theoretical findings. Finally, future work may consider weaker or more generalized contraction conditions, further broadening the scope of fixed point theory.

Conclusion

This study has developed new fixed point results in generalized b-metric spaces under flexible contraction conditions. The proposed theorems extend classical principles by

Online ISSN 3007-3197
Print ISSN 3007-3189

incorporating max-type, rational-type, and integral-type contractions, thereby broadening the scope of fixed point theory. In addition, the extension to multivalued mappings further enhances the applicability of the results.

A key contribution of this work is the formulation of a unified contraction framework, expressed as:

$$\Psi\big(\mathcal{D}(Tx, Ty)\big) \leq k \cdot \Psi\big(M(x, y)\big)$$

where:

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

This unified model captures various contraction types within a single structure, providing a more general and flexible approach to fixed point analysis.

The theoretical findings were supported through applications to integral, differential, and functional equations, confirming their practical relevance. Overall, the results contribute to the advancement of fixed point theory and open new directions for further research in generalized and applied mathematical frameworks.

References:

AGYINGI, C. A. & GABA, Y. U. 2019. q λ-hyperconvexity in quasi-metric spaces. *Afrika Matematika,* 30**,** 399-412.

ALSAHLI, G., KARAPINAR, E. & SHAHI, P. 2025. On rational contractions in perturbed metric spaces. *Research in Mathematics,* 12**,** 2508578.

BANACH, S. 1922. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta mathematicae,* 3**,** 133-181.

BONAB, S. H., PARVANEH, V., BAGHERI, Z. & SHAHKOOHI, R. J. 2023. Extended Interpolative Hardy-Rogers-Geraghty-Wardowski Contractions and an Application. *Advances in Number Theory and Applied Analysis***,** 261-277.

CHATTERJEA, S. K. 1972. Operational representation for the ultraspherical polynomials. *Publikacije Elektrotehničkog fakulteta. Serija Matematika i fizika***,** 37-41.

CVETKOVIĆ, M. 2023. Comparison of F-contraction and φ-contractions. *Filomat,* 37**,** 3951-3961.

CZERWIK, S. 1993. Contraction mappings in $ b $-metric spaces. *Acta mathematica et informatica universitatis ostraviensis,* 1**,** 5-11.

DELFOUR, M. C. 2015. Metric Spaces of Shapes and Geometries Constructed from Set Parametrized Functions. *New Trends in Shape Optimization.* Springer.

DHANUSH, A. & VIJAYAKUMAR, V. 2026. Time-Optimal Control of Hilfer Fractional Stochastic Hemivariational Inequalities Analysis. *Journal of Optimization Theory and Applications,* 208**,** 98.

GRANAS, A. & DUGUNDJI, J. 2003. *Fixed point theory,* Springer.

HARDY, G. E. & ROGERS, T. D. 1973. A generalization of a fixed point theorem of Reich. *Canadian Mathematical Bulletin,* 16**,** 201-206.

Online ISSN 3007-3197
Print ISSN 3007-3189

JAMALI, K., SOLANGI, M. A. & QURESHI, S. 2022. A novel hybrid iterative method for applied mathematical models with time-efficiency. *Journal of Applied Mathematics and Computational Mechanics,* 21**,** 19-29.

JLELI, M. & SAMET, B. 2015. A generalized metric space and related fixed point theorems. *Fixed point theory and Applications,* 2015**,** 61.

KACZYNSKI, T. 2008. Multivalued maps as a tool in modeling and rigorous numerics. *Journal of Fixed Point Theory and Applications,* 4**,** 151-176.

KANNAN, R. 1968. Some results on fixed points. *Bull. Cal. Math. Soc.*, 60**,** 71-76.

KIRK, W. A. Geodesic geometry and fixed point theory. Seminar of mathematical analysis (Malaga/Seville, 2002/2003), 2003. 195-225.

KOLOBOV, V. I., REICH, S. & ZALAS, R. 2017. Weak, strong, and linear convergence of a double-layer fixed point algorithm. *SIAM Journal on Optimization,* 27**,** 1431-1458.

LOHMILLER, W. & SLOTINE*, J.-J. 2005. Contraction analysis of non-linear distributed systems. *International Journal of control,* 78**,** 678-688.

MALIK JASIM, N. & K. ABDALRAHEM, M. 2025. Employing Fixed-Point Theory for Fuzzy Regression Analysis: Methodology and Empirical Application. *F1000Research,* 14**,** 1435.

PANDA, S. K., VIJAYAKUMAR, V. & AGARWAL, R. P. 2025. Generalized Spaces: A Metric Evolution. *Fixed Point Theory from Early Foundations to Contemporary Challenges.* Springer.

SINGH, N. K. & GHOSH, S. 2025. Common fixed point theorems on extended b-metric space using rational-type contraction. *Fixed Point Theory and Algorithms for Sciences and Engineering,* 2025**,** 22.

SONTAG, E. D., MARGALIOT, M. & TULLER, T. On three generalizations of contraction. 53rd IEEE Conference on Decision and Control, 2014. IEEE, 1539-1544.

ZASLAVSKI, A. J. 2016. *Approximate solutions of common fixed-point problems*, Springer.