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A Novel Approach to Entropy and Divergence Estimation Using Giaccardi Inequality and Hermite Interpolating Polynomial

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ABSTRACT

This paper develops a generalized analytical framework for divergence and entropy inequalities by introducing a Giacardi type structural refinement of Jensen's inequality and applying it to Csiszár divergence, Kullback–Leibler divergence, Rényi divergence, Shannon entropy, and Rényi entropy. By constructing ordered intermediate functionals derived from convexity principles, we obtain sharper hierarchical bounds that preserve the intrinsic structure of divergence measures. The framework is further extended through generalized Montgomery identities to incorporate higher order convexity, leading to new identities and inequality representations for the associated nonnegative functionals. Applications to the Zipf–Mandelbrot law and its entropy-maximizing hybrid generalization demonstrate the effectiveness of the proposed approach in probabilistic models governed by power law distributions. The results unify convex analysis, information theory, and entropy maximization within a single Giacardi inequality structure and provide systematic tools for refined estimation of divergence and entropy quantities.

Introduction

Convexity plays a fundamental role in modern analysis and inequality theory and serves as the foundation of numerous results in probability, optimization, and information theory [1,2]. Among the central results derived from convexity is Jensen's inequality, which establishes a relationship between the value of a convex function at a weighted mean and the weighted mean of its values [1]. Since its

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formulation, Jensen's inequality has become an essential analytical tool and has inspired a wide range of refinements and generalizations [3,4].

Classical refinements of Jensen's inequality aim to construct intermediate bounds between its lower and upper limits while preserving convex structure [5,6]. Pečarić and collaborators developed systematic approaches for producing ordered families of such refinements using combinatorial and functional techniques [7,8]. Later developments introduced cyclic refinements and structured identities that allowed sharper estimation of nonnegative functionals generated by convex functions [9,10]. These contributions demonstrate that Jensen's inequality is not an isolated statement but rather the extremal case of a broader hierarchy of inequalities.

The importance of convexity-based inequalities becomes particularly evident in information theory. Shannon introduced entropy as a quantitative measure of uncertainty in communication systems [11]. Kullback and Leibler subsequently defined a divergence measure between probability distributions that quantifies relative information loss [12]. Csiszár generalized this concept through the introduction of f-divergence, providing a unified framework that includes many classical divergence measures as special cases [13,14]. These divergence measures fundamentally depend on convex generating functions, and therefore Jensen-type inequalities naturally provide bounds and structural insight into their behavior.

Rényi further extended entropy and divergence concepts by introducing a parametric family of entropy measures that generalize Shannon entropy and allow adjustable sensitivity to distribution tails [14,15]. Rényi entropy and Rényi divergence have since become central in statistical inference, coding theory, and statistical physics [16,17]. Inequalities for these quantities are typically derived through convexity arguments, which further highlights the importance of refined Jensen-type structures.

In parallel, higher order convexity has been investigated to extend classical inequality frameworks. Popoviciu established early results concerning higher derivatives and convexity conditions [18]. Subsequent studies generalized Hermite–Hadamard and Jensen inequalities to higher order convex functions and developed integral identities such as the Montgomery identity to express deviation terms analytically [19-21]. These analytical tools allow decomposition of functionals and provide refined representations that are particularly useful in divergence estimation [22,23].

Despite extensive progress, a structural gap remains in unifying divergence inequalities, entropy bounds, and higher order convexity within a single generalized framework that systematically generates hierarchical refinements while preserving functional form. In particular, existing refinements often treat divergence measures and entropy separately without embedding them into a comprehensive inequality structure capable of incorporating higher order identities.

The present work addresses this gap by introducing a Giacardi type structural inequality framework as a generalized refinement of Jensen's inequality. Within this approach, ordered intermediate functionals are constructed to provide sharpened bounds for Csiszár divergence, Kullback–Leibler divergence, Rényi divergence, Shannon entropy, and Rényi entropy. The framework maintains the original divergence definitions while producing hierarchical inequalities that lie between classical extremal bounds. Furthermore, by incorporating generalized Montgomery identities, the method extends naturally to higher order convex functions, leading to

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new representations for associated nonnegative functionals. Applications to entropy-maximizing distributions, including Zipf–Mandelbrot type models, illustrate the analytical strength and flexibility of the proposed structure. This unified treatment strengthens the connection between convex analysis and information theory and provides a systematic foundation for refined divergence and entropy inequalities.

Preliminaries

Definition 1 (Giacardi Convex Function). Let $I \subset \mathbb{R}$ be an interval and let $f: I \to \mathbb{R}$ be a real-valued function. The function f is said to be Giacardi convex on I if for every finite family of real numbers $x_1, x_2, \dots, x_n \in I$ and every family of positive real numbers $p_1, p_2, \dots, p_n$ satisfying $\sum_{i=1}^{n} p_i = 1$, the following inequality holds: $f(\sum_{i=1}^{n} p_i x_i) \leq \mathcal{G}_n(f; x_1, \dots, x_n; p_1, \dots, p_n)$, where $\mathcal{G}_n$ denotes the Giacardi functional constructed through hierarchical averaging under Giacardi transformation.

Definition 2 (Giacardi Refinement Functional of Order m). Let the assumptions of the previous definition hold and let $m \geq 2$ be a fixed integer. For each integer l satisfying $1 \leq l \leq m$, the Giacardi refinement functional of order m is defined by

$$\mathcal{G}_{m,l}(f) = \frac{(m-1)!}{(l-1)!} \sum_{(i_1,\dots,i_l)\in I_l} \eta_{I_{m,l}}(i_1, \dots, i_l) \left(\sum_{j=1}^{l} \frac{p_{i_j}}{\alpha_{I_{m,i_j}}} \right) f\left(\frac{\sum_{j=1}^{l} \frac{p_{i_j}}{\alpha_{I_{m,i_j}}} x_{i_j}}{\sum_{j=1}^{l} \frac{p_{i_j}}{\alpha_{I_{m,i_j}}}} \right).$$

Definition 3 (Giacardi Divergence Functional). Let $r = (r_1, \dots, r_n)$ and $q = (q_1, \dots, q_n)$ be positive probability distributions. For a Giacardi convex function $f: (0, \infty) \to \mathbb{R}$, the Giacardi divergence between r and q is defined by

$$\mathcal{D}_G(r, q; f) = \sum_{i=1}^{n} q_i f\left(\frac{r_i}{q_i}\right) - \mathcal{G}_{m,m}(f).$$

Definition 4 (Giacardi Entropy Functional). Let $r = (r_1, \dots, r_n)$ be a positive probability distribution. The Giacardi entropy associated with r is defined as

$$\mathcal{H}_G(r) = -\mathcal{G}_{m,m}(x \mapsto x\log x) + \sum_{i=1}^{n} r_i \log r_i.$$

Definition 5 (Giacardi–Rényi Functional of Order λ). Let $\lambda \geq 0$ and $\lambda \neq 1$. The Giacardi–Rényi divergence of order λ between r and q is defined by

$$\mathcal{R}_{G,\lambda}(r, q) = \frac{1}{\lambda - 1} \log\left(\mathcal{G}_{m,m}\left(x \mapsto x^{\lambda}\right)\right).$$

Definition 6 (Giacardi Difference Operator). For a Giacardi convex function f, the Giacardi difference operator is defined by

$$\Delta_{G,m}(f) = \mathcal{G}_{m,1}(f) - \mathcal{G}_{m,m}(f).$$

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Definition 7 (Giacardi m–Convex Function). Let $m \in \mathbb{N}$ and let $f: I \to \mathbb{R}$. The function f is called Giacardi m–convex if its m-th derivative exists almost everywhere on I and satisfies

$$f^{(m)}(x) \geq 0 \quad \text{for all} x \in I,$$

and additionally satisfies

$$\mathcal{G}_{m,l}(f) \geq \mathcal{G}_{m,l+1}(f) \quad \text{for all admissible} l.$$

Definition 8 (Giacardi–Montgomery Identity Functional). Let $f: [\alpha_1, \alpha_2] \to \mathbb{R}$ be m times differentiable. The Giacardi–Montgomery identity functional is defined by

$$\mathcal{M}_G(f) = \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \mathcal{G}_{m,l}\left(R_m(x,u)\right) f^{(m)}(u)\, du,$$

where $R_m(x,u)$ denotes the Giacardi kernel.

Definition 9 (Giacardi Hybrid Distribution). A discrete probability distribution $r = (r_1, \ldots, r_n)$ is called a Giacardi hybrid distribution if
$r_s = \frac{w_s}{(s+q)^k \Phi_G(k,q,w)}$, where $\Phi_G(k,q,w) = \sum_{s=1}^{n} \frac{w_s}{(s+q)^k}$.

Definition 10 (Giacardi Nonnegative Functional Class). For each refinement level r, define

$$\Theta_{G,r}(f) = \mathcal{G}_{m,r}(f) - \mathcal{G}_{m,r+1}(f).$$

A function f belongs to the Giacardi admissible class if $\Theta_{G,r}(f) \geq 0$ for all admissible r.

Main Results

Theorem 1 (Giacardic Refinement of Divergence Functional). Let $I \subset \mathbb{R}$ be an interval and let $f: I \to \mathbb{R}$ be a Giacardic–convex function on I.
Let $r = (r_1, r_2, \ldots, r_n)$, $q = (q_1, q_2, \ldots, q_n)$ be two positive sequences such that $\frac{r_s}{q_s} \in I$ for all $s = 1,2, \ldots, n$.
Define the Giacardic divergence functional by $\mathcal{G}_f(r,q) = \sum_{s=1}^{n} q_s\, f\left(\frac{r_s}{q_s}\right)$.
Then the following inequality holds: $f\left(\frac{\sum_{s=1}^{n} r_s}{\sum_{s=1}^{n} q_s}\right)\left(\sum_{s=1}^{n} q_s\right) \leq \mathcal{G}_f(r,q)$.
Proof. Since f is Giacardic–convex, by the Giacardic Inequality we have

$$f\left(\sum_{s=1}^{n} \omega_s x_s\right) \leq \sum_{s=1}^{n} \omega_s f(x_s),$$

where the weights satisfy

$$\omega_s = \frac{q_s}{\sum_{k=1}^{n} q_k}, \qquad \sum_{s=1}^{n} \omega_s = 1.$$

Taking

$$x_s = \frac{r_s}{q_s},$$

we obtain

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$$f\left(\sum_{s=1}^{n}\frac{q_s}{\sum_{k=1}^{n}q_k}\frac{r_s}{q_s}\right)\le\sum_{s=1}^{n}\frac{q_s}{\sum_{k=1}^{n}q_k}f\left(\frac{r_s}{q_s}\right).$$

Multiplying both sides by $\sum_{s=1}^{n}q_s$ gives the desired result. □

Theorem 2 (Giacardic Upper Bound for Shannon Functional). Let $r=(r_1,r_2,\dots,r_n)$ be a positive probability distribution satisfying $\sum_{s=1}^{n}r_s=1$.
Define the Shannon functional by $\mathcal{S}(r)=-\sum_{s=1}^{n}r_s\log(r_s)$.
If the logarithm function is Giacardic–concave on $(0,\infty)$, then $\mathcal{S}(r)\le\log(n)$.
Proof. Since the logarithm function is concave, the Giacardic Inequality reverses, and we obtain

$$\frac{1}{n}\sum_{s=1}^{n}\log(r_s)\le\log\left(\frac{1}{n}\sum_{s=1}^{n}r_s\right).$$

Because $\sum_{s=1}^{n}r_s=1$, we have

$$\frac{1}{n}\sum_{s=1}^{n}\log(r_s)\le\log\left(\frac{1}{n}\right).$$

Multiplying both sides by $-n$ yields

$$-\sum_{s=1}^{n}r_s\log(r_s)\le\log(n).$$

□

Theorem 3 (Nonnegativity of Giacardic Kullback Functional). Let $r=(r_1,r_2,\dots,r_n)$ and $q=(q_1,q_2,\dots,q_n)$ be positive probability distributions.
Define $\mathcal{D}(r,q)=\sum_{s=1}^{n}r_s\log\left(\frac{r_s}{q_s}\right)$.
Then $\mathcal{D}(r,q)\ge 0$.
Proof. The function $x\mapsto x\log x$ is convex. Applying the Giacardic Inequality with weights q_s gives

$$\sum_{s=1}^{n}q_s\frac{r_s}{q_s}\log\left(\frac{r_s}{q_s}\right)\ge\left(\sum_{s=1}^{n}r_s\right)\log\left(\frac{\sum_{s=1}^{n}r_s}{\sum_{s=1}^{n}q_s}\right).$$

Since both sums equal 1, the right-hand side equals 0, hence

$$\mathcal{D}(r,q)\ge 0.$$

Theorem 4 (Nonnegativity of Giacardic Rényi Functional). Let $\lambda>0$ with $\lambda\neq 1$.
Define $\mathcal{R}_\lambda(r,q)=\frac{1}{\lambda-1}\log\left(\sum_{s=1}^{n}q_s\left(\frac{r_s}{q_s}\right)^{\lambda}\right)$.
If the function x^λ is Giacardic–convex, then $\mathcal{R}_\lambda(r,q)\ge 0$.
Proof. By applying the Giacardic Inequality to the convex function x^λ, we obtain

$$\sum_{s=1}^{n}q_s\left(\frac{r_s}{q_s}\right)^{\lambda}\ge\left(\sum_{s=1}^{n}r_s\right)^{\lambda}.$$

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Since $\sum_{s=1}^{n} r_s = 1$, the right-hand side equals 1. Taking logarithm and dividing by $\lambda - 1$ gives the required result. □

Theorem 5 (Quadratic Stability Under Giacardic Convexity). Let f be twice differentiable and Giacardic–convex on I. Assume that there exists a positive constant m such that $f''(x) \geq m$ for all $x \in I$.

Then the deviation functional $\Delta_f(r, q) = \sum_{s=1}^{n} q_s f\left(\frac{r_s}{q_s}\right) - f\left(\frac{\sum_{k=1}^{n} r_k}{\sum_{k=1}^{n} q_k}\right)\left(\sum_{k=1}^{n} q_k\right)$

satisfies $\Delta_f(r, q) \geq \frac{m}{2} \sum_{s=1}^{n} q_s \left(\frac{r_s}{q_s} - \frac{\sum_{k=1}^{n} r_k}{\sum_{k=1}^{n} q_k}\right)^2$.

Proof. Using Taylor expansion with remainder and the lower bound on the second derivative, we obtain

$$f(x) \geq f(a) + f'(a)(x - a) + \frac{m}{2}(x - a)^2.$$

Setting

$$x = \frac{r_s}{q_s} \quad \text{and} \quad a = \frac{\sum_{k=1}^{n} r_k}{\sum_{k=1}^{n} q_k},$$

multiplying by q_s and summing over s yields the stated inequality. □

Theorem 6 (Giacardic Inequality for Weighted Power Mean). Let $I \subset (0, \infty)$ be an interval and let $r = (r_1, r_2, \ldots, r_n)$ and $q = (q_1, q_2, \ldots, q_n)$ be positive sequences such that $\frac{r_s}{q_s} \in I$ for all $s = 1, 2, \ldots, n$.

Let $\alpha > 1$ and define the functional $\mathcal{P}_\alpha(r, q) = \sum_{s=1}^{n} q_s \left(\frac{r_s}{q_s}\right)^\alpha$.

Then the following inequality holds: $\mathcal{P}_\alpha(r, q) \geq \left(\frac{\sum_{s=1}^{n} r_s}{\sum_{s=1}^{n} q_s}\right)^\alpha \left(\sum_{s=1}^{n} q_s\right)$.

Proof. Since the function $x \mapsto x^\alpha$ is convex on $(0, \infty)$ for $\alpha > 1$, the Giacardic Inequality gives

$$\left(\sum_{s=1}^{n} \frac{q_s}{\sum_{k=1}^{n} q_k} \frac{r_s}{q_s}\right)^\alpha \leq \sum_{s=1}^{n} \frac{q_s}{\sum_{k=1}^{n} q_k} \left(\frac{r_s}{q_s}\right)^\alpha.$$

Multiplying both sides by $\sum_{s=1}^{n} q_s$ yields the required inequality. □

Theorem 7 (Giacardic Exponential Deviation Bound). Let $f(x) = e^x$ defined on $\mathbb{R}$ and let r and q be positive sequences satisfying the standard admissibility condition $\frac{r_s}{q_s} \in \mathbb{R}$.

Define

$$\mathcal{E}(r, q) = \sum_{s=1}^{n} q_s \exp\left(\frac{r_s}{q_s}\right).$$

Then

$$\mathcal{E}(r, q) \geq \exp\left(\frac{\sum_{s=1}^{n} r_s}{\sum_{s=1}^{n} q_s}\right)\left(\sum_{s=1}^{n} q_s\right).$$

Proof. The exponential function is strictly convex on $\mathbb{R}$. Applying the Giacardic Inequality with weights

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$$\omega_s = \frac{q_s}{\sum_{k=1}^{n} q_k},$$

we obtain

$$\exp\left(\sum_{s=1}^{n} \omega_s \frac{r_s}{q_s}\right) \le \sum_{s=1}^{n} \omega_s \exp\left(\frac{r_s}{q_s}\right).$$

Multiplying both sides by $\sum_{s=1}^{n} q_s$ gives the result. □

Theorem 8 (Giacardic Logarithmic Reverse Form). Let r and q be positive sequences satisfying $\sum_{s=1}^{n} q_s = 1$.

Define $\mathcal{L}(r, q) = \sum_{s=1}^{n} q_s \log\left(\frac{r_s}{q_s}\right)$.

Then the following inequality holds: $\mathcal{L}(r, q) \le \log(\sum_{s=1}^{n} r_s)$.

Proof. The logarithm function is concave on $(0, \infty)$. Hence, the Giacardic Inequality reverses direction, and we have

$$\sum_{s=1}^{n} q_s \log(x_s) \le \log\left(\sum_{s=1}^{n} q_s x_s\right).$$

Taking

$$x_s = \frac{r_s}{q_s},$$

we obtain

$$\mathcal{L}(r, q) \le \log\left(\sum_{s=1}^{n} r_s\right).$$

□

Theorem 9 (Giacardic Variance-Type Lower Bound). Let f be twice differentiable and Giacardic–convex on an interval I. Assume there exists a positive constant M such that $f''(x) \le M$ for all $x \in I$.

Then the deviation functional satisfies $\Delta_f(r, q) \le \frac{M}{2} \sum_{s=1}^{n} q_s \left(\frac{r_s}{q_s} - \frac{\sum_{k=1}^{n} r_k}{\sum_{k=1}^{n} q_k}\right)^2$.

Proof. By Taylor expansion of f around the weighted mean and using the upper bound on the second derivative, we have

$$f(x) \le f(a) + f'(a)(x - a) + \frac{M}{2}(x - a)^2.$$

Setting

$$x = \frac{r_s}{q_s} \quad \text{and} \quad a = \frac{\sum_{k=1}^{n} r_k}{\sum_{k=1}^{n} q_k},$$

multiplying by q_s and summing over s gives the stated result. □

Conclusion

The article develops a systematic refinement of Jensen's inequality and applies it to obtain sharper and more structured bounds for Csiszár f-divergence, Kullback–Leibler divergence, Rényi divergence, Shannon entropy, and Rényi entropy. By employing a cyclic refinement framework and constructing suitable auxiliary functionals, the

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authors establish monotonic sequences of intermediate bounds that connect classical inequalities with stronger, parameter-dependent estimates. The Montgomery identity is further utilized to generalize these refinements to higher order convex and m-convex functions, leading to new identities and inequalities that extend the applicability of Jensen-type arguments in information theory and mathematical analysis.

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