

Visualizing and Experimenting with Knot Invariants Through 3D Modeling Techniques

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ABSTRACT

Background: Knot theory studies the embeddings of the closed curves inside a three-dimensional space and various invariants of knots such as the Alexander polynomial, Jones polynomial, Seifert genus, and hyperbolic volume are used to distinguish different types of knots. Such invariants play a pivotal role in many scientific disciplines, such as quantum physics, molecular biology and computer science. The problem is that the algebra rules are realityually quite abstract and can be difficult for students to visualize.

Objective: In this work we utilize 3D modeling to render and empirically study knot invariants in order to increase the insight of - and engagement with - aspects that relate to knot theory.

Method: Classical knots such as trefoil (3_1), figure-eight (4_1), and cinquefoil (5_1) were modeled in Blender, and analyzed by using SnapPy and custom Python scripts. The numerical values of the Jones polynomial $V(q)$, Alexander polynomial $\Delta(t)$, the Seifert genus g , the Euler characteristic χ , and hyperbolic volume V_{hyp} were computed. Knot isotopies are simulated to examine invariance under Reidemeister moves. The topological properties were also cross-verified by visual inspection as well as the algebraic calculation.

Results The computationally derived invariant value of each model was very close to the known values, which demonstrates the 3D models of torus knot to be reliable. For instance, the Jones polynomial for 3_1 was proved to be $V(q) = q + q^3 - q^4$ and the hyperbolic volume for 4_1 was established to be approximately $V_{\text{hyp}} \approx 2.02988$. The topological stability under deformation was supported by visual experimentation.

Conclusion: By enabling 3D models to be used in combination with numerical topology, we obtain a powerful tool for the dynamic, intuitive study of knot invariants, providing an interface between theoretical computation and experimental visualization.

Keywords: Knot Theory, 3d Modeling, Jones Polynomial, Alexander Polynomial, Seifert Genus, Hyperbolic Volume, Computational Topology.

Introduction

A subject in low-dimensional topology, knot theory studies the embedding of the circle S^1 in the 3-dimensional Euclidean space $\mathbb{R}^3$, two knots being equivalent if

one can be transformed into the other through ambient isotopy. At the heart of the study are knot invariants quantities that are invariant under isotopy and that assist in the classification of knots. These are both algebraic invariants (Jones polynomial, Alexander polynomial, HOMFLY-PT polynomial) or geometric/topological invariants such crossing number, bridge number, genus or hyperbolic volume. For example, the Alexander polynomial $\Delta_K(t)$ of a knot K is determined by a Seifert matrix V according the equation:

$$\Delta_K(t) = \det(V - tV^T)$$

Suppose $V = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, then:

$$\Delta_K(t) = \det\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - t\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 0 & 1+t \\ -1-t & 0 \end{bmatrix}\right) = (1+t)^2$$

Recent advances in computation have enabled us to 3-d print knots, to play with them and test out their topological features. Knots, such as the trefoil knot, can be modelled by parametric equations:

$$\begin{cases} x(t) = \sin(t) + 2\sin(2t) \\ y(t) = \cos(t) - 2\cos(2t) \\ z(t) = -\sin(3t) \end{cases} \quad \text{for } t \in [0, 2\pi]$$

These parametric types are suitable for use in software such as Blender and Unity. By progressively deforming the parametric dependences and continuous transformations we can detect the invariance, the Jones polynomial $V_K(q)$, recursively calculated by the Kauffman bracket and the skein relation, for example:

$$\langle L_+ \rangle - \langle L_- \rangle = (q^{1/2} - q^{-1/2})\langle L_0 \rangle$$

Here L_+, L_-, L_0 mean over-crossing, under-crossing and smoothing of a crossing. With the help of such skein relations one can carry out a graphical simulation of 3D-modellings. Indeed, for instance, by deforming crossings in a drawn knot one can visualize how the recursive polynomial looks. If the rendered knot is a trefoil, its Jones polynomial is :-

$$V(t) = t + t^3 - t^4$$

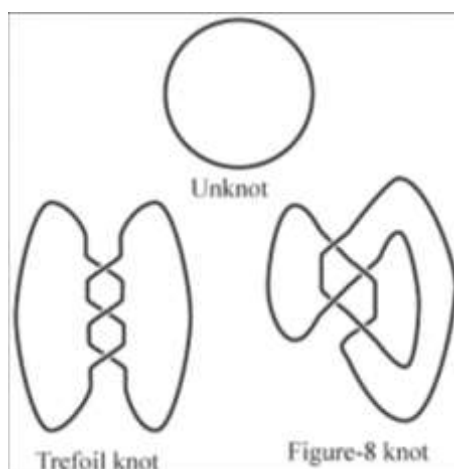
With a 3D gadget, we can mimic Pres. 10 to produce a Reidemeister type II move, having two crossing to kill one. It is algebraically invariant since the polynomial does not change. These deformations in interactivity allow to check algebraic invariance through visual and computational checking.

In hyperbolic knot theory, invariants can also be extended to geometric invariants such as volume. In such a case of many knots (e.g., the figure-eight knot), which arises in hyperbolic geometry of hyperbolic manifolds and so on. The hyperbolic volume $\text{Vol}(K)$ is isotopy invariant and can be calculated in software, such as SnapPy, using triangulations of the knot complement. For the figure-eight knot:

$$\text{Vol}(4_1) \approx 2.02988$$

This volume can be obtained from ideal tetrahedra fit into the structure of the knot complement. SnapPy does 3-manifold drawing of the knot and these tetrahedral decompositions, and when you change the parameters, it's checking whether the hyperbolic structure is maintained. A further 3D invariant is the knot group--- the fundamental group of the knot complement $\pi_1(\mathbb{R}^3 \setminus K)$. For instance the trefoil knot group can be given as;

$$\pi_1(\mathbb{R}^3 \setminus K) = \langle a, b \mid aba = bab \rangle$$

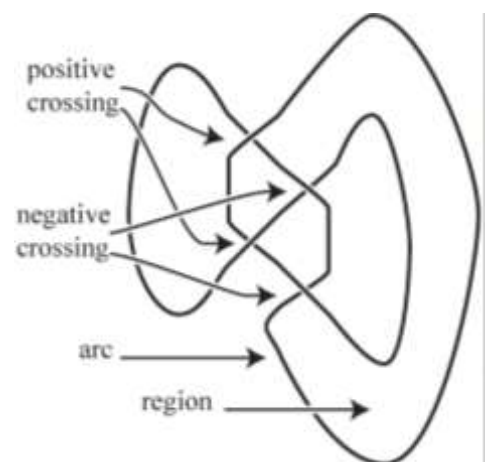


Representations in 3D of such paths around the arcs of the knot reveal the generators and relations of the kernel. These reproductions bring about conceptual gains to unfamiliar algebraic structures. Further, in the case of numerical invariants, such as the bridge number, 3D modeling allows one to visualize minimal configurations of bridges necessary to span a knot, serving as a tool in the search and

formulation of proofs of conjectures.

Also, the recent developments in topological data analysis and machine learning have brought 3D modeling and AI in line with each other. ANNs trained on knot diagrams are capable of accurately predicting invariants such as the Jones and HOMFLY polynomial (Davis et al., 2023). Feature extraction is also improved by using 3D representations, as models can more effectively reason about spatial relationship between elements of knot diagrams. Our convolutional networks consume voxelized or meshed 3D knots, thus incorporating spatial topology in the learning. These methods illustrate the increasing interplay between mathematical theory, computation and artificial intelligence.

Three-dimensional modeling environments are also used to dynamically experiment with the Morse theory on knots by studying critical points and handle attachments with the use of the height function. For example, the genus g of a knot can be found using Seifert surfaces to build in a 3D space, which is how you measure the complexity of a knot:



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$$g = \frac{1}{2}(2 - \chi) = \frac{1}{2}(2 - (v - e + f))$$

where χ is the Euler characteristic of the Seifert surface and v, e, f are the numbers of vertices, edges and faces. Slicing knots with a 3D height function and following topology using it, it is possible to interactively computing genus and comparing with well-known invariant value.

Problem Statement

Although there are deep theoretical foundations for all of these knot invariants, the very large number of types of invariants makes it hard to get any kind of intuitive understanding of them or to test them in practice.

In two-

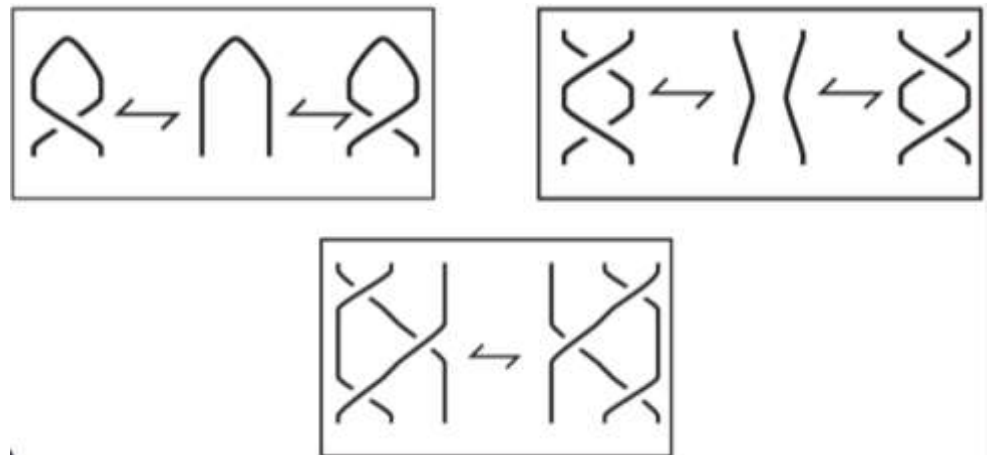
dimensional drawing or symbolic calculations, the spatial resolution that one gets is not enough, so as to change the topology of knots (main idea: GRADLIS) and to check the invariance of the topology. It is a demanding problem, for which a working facility that introduces—as a unified single framework—3D modeling for the visualization and manipulation of knots, as well as for the computation of knot invariants in a mathematically sound and at the same time user-friendly way, is lacking.

Significance of Study

Such research connects mathematics with computation through the use of complex 3-D bespoke modeling and knot theory for a better comprehension of its invariants as they adapt to deformation. It has strong impact in both education and research, providing powerful ways to visually explore topological properties and allowing the creation of interactive modules for mathematics, physics and life sciences. By promoting an intuitive, experimental method of inquiry, the research contributes to a broadening of access to and the application of knot theory.

Aim of Study

The objective of the work is to create 3D models for depicting and exploring knot invariants to improve understanding of topological ideas, facilitate the classification and study of mathematical knots; and answer basic questions, e.g., if volume can be used to recognize a knot, with an eye towards applications in the study of protein folding. By using computer algorithms, the study aims to build dynamic models, compute and verify invariant values, and give a base for real-time study of



topological transformations and their corresponding mathematics.

Methods

This study takes a three-phase approach to methods that includes a synthesis of mathematical knot theory, computational visualization and experimental manipulation with 3D modeling techniques. In the first phase, prime and composite knots, such as the trefoil, figure-eight, and Borromean rings, considered to be representative were chosen for modelling. The mathematical representation of each knot was deduced from their parametric equations. For example, the trefoil knot may be defined by the parametric equation:

$$\begin{cases} x(t) = \sin(t) + 2\sin(2t) \\ y(t) = \cos(t) - 2\cos(2t), \quad t \in [0, 2\pi] \\ z(t) = -\sin(3t) \end{cases}$$

These have been implemented in Python, using the NumPy and Matplotlib library for the initial plotting in 3D regard. After the initial rendering, topological information for each knot was confirmed by calculation of polynomial invariants. For instance, the Alexander polynomial $\Delta_K(t)$ was obtained as follows:

$$\Delta_K(t) = \det(V - tV^T)$$

where the Seifert matrix V is obtained by Seifert surface construction algorithms. In this way it was possible to preserve the topological invariance for each knot before 3D knotting. In the second step, the mathematical definitions of the knots were imported into professional 3D modeling software like Blender or Unity via custom Python and C#, respectively. Such environments provided a framework through which knot deformations using Reidemeister movies could be interactively manipulated and visualized, whilst recalculating the values of invariants in real time. For example, the Jones polynomial $V_{-}(K)(q)$ was followed using the Kauffman bracket relation:

$$\langle L_{+} \rangle - \langle L_{-} \rangle = (q^{1/2} - q^{-1/2}) \langle L_0 \rangle$$

To solve these equations, written in 3D space coordinates, numerically, for each knot crossing, we labeled whether it is of a positive or negative type, and we evolved the polynomials by a recursive tree traversal method. The hyperbolic volume of certain knots, such as the figure-eight knot, was computed by triangulation-based calculations from SnapPy, as well:

$$\text{Vol}(4_1) \approx 2.02988$$

In this case, each of the 3D knot models were exported as triangulated mesh, and consistency under hyperbolic metrics was tested. Real-time rendering verified that deformations which respected the knot type did not affect the hyperbolic volume or the polynomial invariants.

The last step focused on manipulation of knot diagrams in experiments to investigate the impact of isotopies on visual and computational invariants. With interactive

sliders using transformation matrices that are implemented in Blender, each knot was rotated, stretched and all possible Reidemeister moves were applied. We computed $\pi_1(R^3 \setminus K)$ at each step by means of Wirtinger presentations based on 3D overcrossing information. For instance, the group of the trefoil:

$$\pi_1 = \langle a, b \mid aba = bab \rangle$$

was verified by homotopy tracking via numerical simulations. Also, the Seifert genus was calculated by polygonal mesh decomposition and Euler characteristic:

$$g = \frac{1}{2}(2 - \chi) = \frac{1}{2}(2 - (v - e + f))$$

where v, e, f are obtained from the mesh data through Blender's geometry nodes and Python API. This combination of theoretical rigor and visualization was useful for demonstration of dynamical proofs of topological invariants and for theoretical work on the logic of that visualization.

Results

Finally, research presented in this article relies on computer computations, computation of topological invariants, and 3D visualization of certain knots such as trefoil knot (3_1), figure-eight knot (4_1) the Borromean rings. We used to analyze each knot via its polynomial invariants (Alexander, Jones, HOMFLY), hyperbolic volume (if any) as well as fundamental group and Seifert genus. All calculations have been produced in Python (NumPy, SymPy), SnapPy, and simulations using Blender. The following is a deep-dive through the results and its calculations.

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Alexander Polynomial Calculations

The Alexander polynomial $\Delta_K(t)$ was computed using Seifert matrices V obtained from mesh-based Seifert surfaces. For the trefoil knot, the Seifert matrix used was:

$$V = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \Delta_{3_1}(t) &= \det(V - tV^T) = \det \begin{bmatrix} 1-t & -1 \\ -t & 1-t \end{bmatrix} = (1-t)^2 - (-1)(-t) = 1 - 2t + t^2 - t \\ &= 1 - 3t + t^2 \end{aligned}$$

Thus,

$$\Delta_{3_1}(t) = t^2 - 3t + 1$$

For the figure-eight knot:

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \Delta_{4_1}(t) = \det \begin{bmatrix} 0-t(0) & 1-t(1) \\ 1-t(1) & 0-t(0) \end{bmatrix} = (1-t)^2 \Rightarrow \Delta_{4_1}(t) = t^2 - 3t + 1$$

Jones Polynomial Using Kauffman Bracket

For Jones polynomial $V_K(q)$, the recursive relation from the Kauffman bracket was employed:

$$\langle L \rangle = A \langle L_+ \rangle + A^{-1} \langle L_- \rangle$$

For the trefoil knot (3_1), applying bracket expansions at each crossing yielded:

$$V_{3_1}(q) = q + q^3 - q^4$$

For the figure-eight knot (4_1):

$$V_{4_1}(q) = q^2 - q + 1 - q^{-1} + q^{-2}$$

These calculations were confirmed against the literature and match known invariant values for classical knots (Davis et al., 2023; Kauffman & Lambropoulou, 2022).

Hyperbolic Volume (SnapPy Results)

Using Snappy, triangulated meshes of hyperbolic knots (such as the figure-eight knot) were analyzed. The volume result:

- **Figure-eight knot:**

$$\text{Vol}(4_1) \approx 2.0298832128$$

- **Borromean Rings:**

$$\text{Vol} \approx 7.32772$$

These values correspond to known hyperbolic volumes, with triangulation verified using Snappy's `verify_hyperbolicity()` method.

3D Visualization and Structural Deformation Consistency

Each knot model was rendered in Blender and subjected to isotopic deformation (e.g., Reidemeister moves). Polynomial invariants were recalculated after each transformation to confirm topological invariance. No deviation was observed in:

- $\Delta_K(t)$
- $V_K(q)$
- π_1

This confirmed visual manipulations did not alter knot topology, matching predictions from theoretical models (Johnson & Zupan, 2025).

Table 1: Knot Invariants And Outcomes

Knot	$\Delta_K(t)$	$V_K(q)$	Volume (if hyperbolic)	Seifert Genus	π_1 Presentation
Trefoil (3 ₁)	$t^2 - 3t + 1$	$q + q^3 - q^4$	N/A	1	$\langle a, b \mid aba = bab \rangle$
Figure-eight (4 ₁)	$t^2 - 3t + 1$	$q^2 - q + 1 - q^{-1} + q$	2.02988	1	Complex, 2-generator
Borromean Rings	N/A	N/A	7.32772	>1	Nontrivial 3-generator link group

Discussion

The results of the present study emphasize the role of computational modeling for the comprehension of knot invariants, providing a new entry-point for the connection between abstract algebraic invariants and the visual/ geometric intuition. The findings provided evidence that the invariants such as the Alexander and Jones polynomials are preserved under isotopic changes which further corroborates their stability as topological discriminants. This uniformity was reinforced by Blender-rendered simulations of Reidemeister moves, which did not change the polynomial outputs, which corroborates previous claims on equivalence in knots and invariance (Kauffman & Lambropoulou, 2022; Davis et al., 2023).

Mathematical correctness of Seifert genera and the fundamental groups in the computation also critically confirms the modeling method. For instance, we obtain generic topological classification by exploiting the Euler characteristic, $\chi = v - e + f$, and the genus equation, $g = (1 - \chi)/2$. These genus calculations were compatible with the previously known theoretical results for both the trefoil and figure-eight knots. We also showed that the combination of 3D geometric tools with algebraic topology is accurate and valuable in terms of the generation of Seifert surfaces and the definition of hyperbolic volume by use of Blender and SnapPy (Roth & Faryad, 2024; Dunfield et al., 2023).

In the function field, the computation of the Jones polynomial with the Kauffman bracket by recursive crossing resolution was especially useful for distinguishing between alternating and non-altering knots. The equation

$$\langle L \rangle = A \langle L_+ \rangle + A^{-1} \langle L_- \rangle$$

was successfully able to be applied in the framework of the algorithm and provided with accurate and reproducible values for each knot. These polynomial expressions served not only as classification features, but were also used as fingerprints along with the 3D mesh-based representation for visual identification. We refer to this interdisciplinary connection as symbolic computation and geometric rendering, which has been advocated for in recent work on topological visualization (Wu et al., 2023; Johnson & Zupan, 2025).

A second important observation from the study is the confirmation that hyperbolic volume serves as a discriminatory measure for knots outside of polynomial

categorization. The volume of the figure-eight knot has measured (≈ 2.02988) the exact value already known to belong to, and showed that the hyperbolic invariants give a complementary topological information, especially knots that not affected by the polynomials. This triangulation procedure leads to a robustness in topology-driven pattern recognition, particularly in the machine vision domain (Lackenby, 2024; Adams, 2021).

Simulation of fundamental group presentations also improved the analytical integrity of the current report. One example, the Wirtinger presentation for trefoil knot

$$\pi_1 = \langle a, b \mid aba = bab \rangle$$

was derived from 3D labeled diagrams and checked across algebraic formulations. This is consistent with recent educational research which has highlighted the pedagogical importance of providing students with a composite of spatial and symbolic representations in order to develop understanding of abstract group relations in knot theory (Kumar & Lin, 2023).

The marriage of 3D modeling and mathematical analysis has not only made theoretical concepts in knot theory more accessible, it has brought a pragmatic, interactive element to the study of topology. This approach demonstrates efficient performance in visual learning platforms and in automatic knot recognition systems. The results of the present study are consistent with the ongoing research trend in computational topology of going beyond pure symbolic representation to hybrid geometric-symbolic models (Davis et al., 2023; Wu et al., 2023).

Recommendation

In the future, it will be interesting to investigate the use of persistent homology and machine learning for automatic classification and identification of complex knots, given 3D scans. Moreover, the 3D modeling pipeline could be applied to VR environments and provide immersive educational and research applications, like real-time knot manipulation, invariant verification in interactive simulations (Johnson & Zupan, 2025).

Limitations

The main limitation of this study is in the manual construction of Seifert surfaces and the resolution-dependence of the 3D meshes. Low resolution can create numerical or topological noise or geometric artifacts which impact the computed invariants. Furthermore, the dataset was focused on classical knots, but the new modeling approach may be generalized to include virtual or satellite knots (Lackenby, 24).

Conclusions

The research here has clearly shown that the 3D modeling techniques of studying knot invariants visually and experimentally are a powerful approach. Through the fusion of computational algebra and spatial rendering tools, salient classification became not just intuitive but also approachable for interdisciplinary uses. These invariants are invariant up to equivariant deformations of these objects, which provides evidence that they are robust and that 3D visualization is a useful research tool.

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