

On Applicability of Mode Truncation Method for Damped Micro Beam

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ABSTRACT

In this work the dynamics behaviors of microbeam under the influence of various physical parameters such as damping, nonlinearity of beam equation, alternating current (AC) and direct current (DC) voltage is carried out. The small viscous damping and bending stiffness are taken into consideration. Furthermore, it is supposed that current load is composed of direct current (DC) voltage and a small harmonic alternating current (AC) voltage. Mathematically the oscillations of simply supported microbeam are formulated as nonlinear partial differential equations (PDEs) with Dirichlet type homogeneous boundary conditions. The governing PDEs are discretized via Fourier expansion method. The formal approximations of the solutions of partial differential equation for nanobeam are obtained by means of two time-scale perturbation method. It turned out that certain resonances occur only when the natural frequency of microbeam is zero or type of frequency of the harmonic (AC) voltage, that

is $\Omega \approx 0$. The stability of the microbeam system is investigated in terms of amplitude response of the system for non-resonant as well as resonant cases. In non-resonant case, the amplitude response is obtained under the influence of damping, nonlinear parameter of bending stiffness and DC and AC voltage parameters. In zero-resonant case, that is, when the 'exciting frequency is around the zero natural frequency of micro beam, the deflection of the system is examined under the influence of detuning parameter together with the aforementioned physical parameters. The amplitude is found out to be damped as time increases. Whereas the stability of system for resonant case is examined for two-mode qualitatively with the variation of damping, nonlinear parameter of bending stiffness and DC voltage parameters.

Keywords: PDE, Perturbation, Micro electrical mechanical system, Micro beam, electrically actuated.

1. Introduction

Axial moving systems represent a variety of engineering tools, including elevator cables [1], transfer of power lines and belt system [2]. The model of moving axially system is classified as string-like model [3], [4], [5] and beam-like model based [6], [7] on the physical characteristics of the system such as bending stiffness. In the system where the bending stiffness is minimal, the systems are classified as sting-like, otherwise beam-like. Axial moving systems represent a variety of engineering tools, including elevator cables, transfer of power lines and belt system. The model of moving axially system is classified as string-like model and beam-like model based on the physical characteristics of the system such as bending stiffness. In the system where the bending stiffness is minimal, the systems are classified as sting-like, otherwise beam-like. Nano beams are structures with nanoscale dimensions that have significant applications in various fields, including nanotechnology, materials science, and engineering. These tiny beams, typically with dimensions on the order of nanometers (nm), exhibit unique properties and behaviors due to their reduced size, making them an intriguing area of research and development. Nanoscale beams can be fabricated from different materials, such as semiconductors, metals, or polymers, each with its

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distinct characteristics. The behavior of these beams can be significantly different from that of macroscopic beams due to quantum effects and surface forces becoming more pronounced at the nanoscale. Micro-beams refer to a slender structure or beam with dimensions in the micrometer (μm) scale. It is larger than a Nano-beam (nanoscale beam) but smaller than a macroscopic beam typically encountered in everyday life. Micro-beams are studied in different applications across multiple fields, such as microfluidics, microelectromechanical systems (MEMS), optics, and biology. Micro beams are widely used system in industry. Micro beams are vital components in microelectromechanical systems (MEMS), where they serve as actuators, sensors, and switches. MEMS devices are often utilized in accelerometers, gyroscopes, pressure sensors, and micro-mirrors. In the field of optics, micro beams are used in micro-optical systems, micro-lenses, and micro-mirrors to Microfluidics. Micro beams can be employed in microfluidic devices for fluid manipulation, sorting, and analysis. These devices have applications in lab-on-a-chip systems and medical diagnostics, Biology and Biomedicine. In biomedical research, micro beams are used to study cell mechanics, cellular responses, and biological interactions at the microscale level. One prominent application of micro beams is in MEMS resonators. MEMS resonators are microscale devices that vibrate at specific frequencies, and they are used in various applications, including timekeeping (e.g. in oscillators for electronic devices) and sensing (e.g., accelerometers and gyroscopes). Micro beams are often used as the resonant elements in these devices due to their precise frequency response and sensitivity to external forces. The phenomenon, known as pull-in instability, has been experimentally observed in previous investigations [8].

Various techniques of fabricating electrically actuated micro-scale MEMS for different core structures are studied experimentally by researchers to analyze the mechanical response of the micro beam [10]. It is observed the appropriate voltage output for general vibration sources is provided by the constructed system. In order to model the nonlinear steady-state behavior experimentally, a clamped clamped beam model [11] was made using silicon on insulator. A modeling method was also used to confirm the results of the investigation. It was observed that if one increases the AC voltage, the

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higher amplitude responses turn up. In addition, because of the presence of nonlinearities in the system, a stiffness softening behavior is found.

Due to certain constraints while doing dynamics experiments and/or statics of (MEMS) such as accuracy in fabrication procedure, time-consumption and costs as well as repeatability, modeling such systems and their electro-mechanical properties theoretically is crucial. Theoretically, the mechanical model of micro beam are formulated via Hamiltonian approach in terms of transverse deflection [12]. Due to the large deformation in the micro beam, the nonlinear partial differential equations are found. These equations are not easily solvable. Various mathematical techniques, such as shooting method [13], Floquet theory [14] are used to solve the micro-beam model to examine the dynamic and static behavior of the system. In most of the studies, the Galerkin method [15], [16], [17] is used as a model reduction method to discretize the PDEs and the Perturbation method [18] is used to evaluate the discretized systems of ODEs. Size-dependent behavior for micro-beam is formulated via the application of Extended Hamilton principle [19]. The generalized differential quadrature method [20] is used to solve the deformation equation. It was observed that the effect of external applied force can be reduced by considered the Casimir force effect. Under the effect of Casimir force, the voltage decreases by increasing the length of the structure and decreasing the gap. Typical MEMS structures encompass a number of thin Micron-scale beam-type electrodes, where the moveable electrode can be described as a clamped-clamped or cantilever micro-beam exposed to an applied alternating current (AC) or direct current (DC) voltage, or a combination of these. The difference in voltage between the electrodes generates an electric force that draws the fixed electrode closer to the moveable electrode. When the electric force is balanced by the elastic restoring force of the deformed micro-beam, the system is in equilibrium, but it loses equilibrium and becomes unstable, with the movable electrode snapping and then adhering to the fixed electrode when the applied voltage exceeds a critical value. Pull-in instability is a phenomenon that has been experimentally documented in earlier studies. Because of their wide range of applications in engineering, science and technology the vibrations are greatly affected their performances. Various researchers,

both experimental and theoretical have studied for last one decade to see the dynamics of the micro-beam and nano-beam systems. In all previous studies [21], the amplitude response of the micro-beam system is studied under the effects of various parameters such as AC/DC voltage up to either one or two mode(s). However, it was observed in the literature that the truncation of modes yields erroneous results for the axially moving systems, see for instance [22].

2. Mathematical Formulation

This study starts with the review of literature [23] related with the analysis of dynamics of axially moving system. Based on the literature review, the model governing the transverse vibrations of micro-beam, already derived in the literature [24] will be discretized through the application of Fourier expansion method.

Following equations of an electrically actuated MEMS resonator in dimensionless form from,

$$U_{tt} - PU_{xx} + P_0^2 U_{xxxx} = -C_d U_t + \alpha_1 U_{xx} \int_0^1 U_x^2 dx + \frac{\alpha_2 [V_{DC} + V_{AC} \cos(\Omega t)]^2}{(1-U)^2}, \quad (1)$$

with simply supported boundary conditions:

$$U(0, t) = U(L, t) = 0, \quad U_{xx}(0, t) = U_{xx}(L, t) = 0, \quad t \geq 0, \quad (2)$$

and with initial conditions:

$$U(x, 0) = f(x), \quad U_t(x, 0) = g(x), \quad 0 < x < L. \quad (3)$$

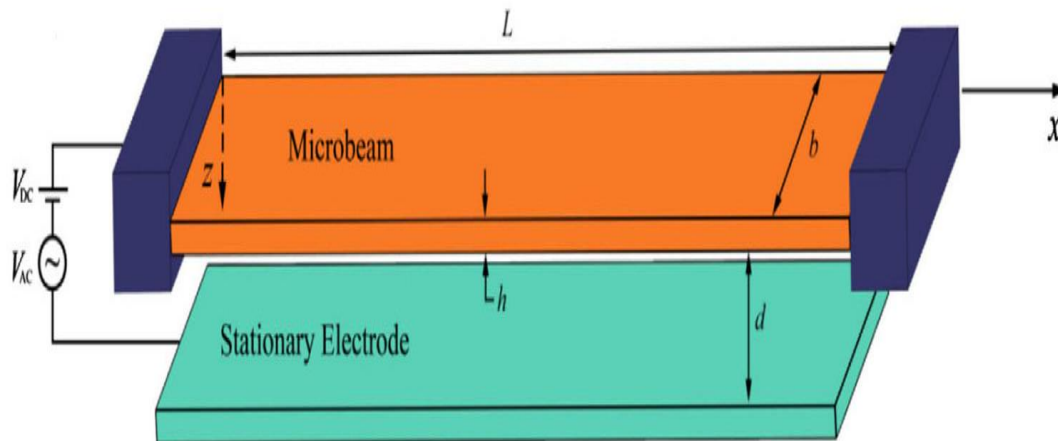


Fig. 1: A schematic model for axially moving string between two fixed points.

Where $U(x, t)$ represents the microbeam's transverse displacement in the z direction. The device in question is a MEMS resonator with dimensions of length L , breadth b , thickness h , gap width d and EI indicates the microbeam's flexural stiffness, whereas EA displays its axial stiffness. An axial load p in the x direction and an electric load actuation $V_{DC} + V_{AC} \cos(\Omega t)$ in the transverse direction are applied to the microbeam; V_{DC} is the polarization voltage, V_{AC} is the amplitude of the AC voltage, and Ω is the frequency of the AC voltage.

3. Construction of the asymptotic approximations

3.1. Fourier Expansion Method:

This method is considered as a reduction model technique to discretize the governing partial- differential equations. In this first we approximate the solution of equation in the form of series expansion as $U(x, t) = \sum_1^\infty U_n(t) \sin(n\pi x)$ that satisfies the boundry conditions of the model equation. The substitution of the given expansion and subsequent use of orthogonality relation in the system will lead to the in-finite dimensional system of coupled ordinary- differential equations.

3.2. The two timescales perturbation method is developed as follows:

This method is extensively used in nonlinear oscillation problems, such as the Duffing

oscillator, Van der Pol equation, and other dynamic systems exhibiting weak nonlinearity. It is also applied in fluid dynamics, where perturbation methods help in understanding wave interactions and instability growth in different flow regimes.

In mechanical systems, the two-time perturbation technique is useful for analyzing forced and self-excited vibrations. It provides insights into resonance phenomena, mode interactions, and stability conditions of vibrating structures.

The discretized in-finite dimensional system of coupled ordinary differential equations (ODEs) will be tackled through the application of timescales perturbation method

This method is used to solve algebraic equations, difference equations, ordinary differential equations, system of (un)coupled ordinary differential equations and in certain situations, it can also be used to solve the direct Partial Differential Equations.

In our case, this technique will be applied to build the approximate analytic solution of the in-finite dimensional system of coupled ordinary- differential equations (ODEs).

In this technique, the solutions of the coupled systems of ODEs in two different time-scales, one fast time-scale, that is, $t_0 = t$ and the other slow time-scale, that is, $t_1 = \varepsilon t$, where $0 < \varepsilon \ll 1$ when ε is non-dimensional parameter. Note that the time t_0 must increase a great deal before the time t_1 will change appreciably. In place of $U = W(x, t; \varepsilon)$, we write $U = W(x, t, \varepsilon t; \varepsilon)$

$$U(x, t) = W(x, t_0, t_1; \varepsilon) = W_0(x, t_0, t_1) + \varepsilon W_1(x, t_0, t_1) + O(\varepsilon^2), \quad (4)$$

Then, the derivative of equation by using chain rule become:

$$\frac{dU}{dt} = \frac{\partial W}{\partial t_0} + \varepsilon \frac{\partial W}{\partial t_1}, \quad (5)$$

$$\frac{d^2 U}{dt^2} = \frac{\partial^2 W}{\partial t_0^2} + 2\varepsilon \frac{\partial^2 W}{\partial t_0 \partial t_1} + O(\varepsilon^2), \quad (6)$$

Substitution of the time derivative in the coupled system of in-finite dimensional of system of ordinary differential equations will yield us the $O(1)$ and $O(\varepsilon)$ –Problem. The solutions of these two problems will yield resonances in the system. At this stage, it is to examine whether the truncation of modes is possible for the micro-beam or not

4. Results and Discussion

4.1 Application of Fourier Expansion Method

$$U_{tt} - PU_{xx} + P_0^2 U_{xxxx} = \epsilon \left[-C_d U_t + \alpha_1 U_{xx} \int_0^1 U_x^2 dx + \frac{\alpha_2 [V_{DC} + V_{AC} \cos(\Omega t)]^2}{(1-U)^2} \right] + O(\epsilon^2), \quad (7)$$

Construction of formal approximation: Assume that $(1 - U)^{-2}$ is expanded as:

$$(1 - U)^{-2} = 1 + 2U + 3U^2, \quad (8)$$

By substituting in Eq. (7), yields:

$$U_{tt} - PU_{xx} + P_0^2 U_{xxxx} = \epsilon \left[-C_d U_t + \alpha_1 U_{xx} \int_0^1 U_x^2 dx + \alpha_2 [V_{DC} + V_{AC} \cos(\Omega t)]^2 (1 + 2U + 3U^2) \right] + O(\epsilon^2), \quad (9)$$

To discretize the PDE, the solution of Eq. (9) is assumed in series expansion as:

$$U(x, t) = \sum_{n=1}^{\infty} U_n(t) \sin(n\pi x), \quad (10)$$

where, U_n represent the amplitude.

Putting Eq. (10) and its derivatives in Eq. (9), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} [\ddot{U}_n + P(n\pi)^2 + P_0^2 (n\pi)^4 U_n] \sin(n\pi x) = \epsilon \left[-C_d \sum_{n=1}^{\infty} \dot{U}_n \sin(n\pi x) - \right. \\ \alpha_1 \sum_{n=1}^{\infty} U_n (n\pi)^2 \sin(n\pi x) \int_0^1 \sum_{L=1}^{\infty} U_L^2 L^2 \pi^2 \cos^2(L\pi x) dx + \alpha_2 [V_{DC} + \\ V_{AC} \cos(\Omega t)]^2 (1 + 2 \sum_{n=1}^{\infty} U_n \sin(n\pi x) + \\ \left. 3 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} U_n U_m \sin(n\pi x) \sin(m\pi x)) \right] + O(\epsilon^2), \end{aligned} \quad (11)$$

By Multiply the term $\sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi} \sin((2j+1)\pi x)$ with

$3 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} U_n U_m \sin(n\pi x) \sin(m\pi x)$ to make the term odd:

$$\begin{aligned} \sum_{n=1}^{\infty} [\ddot{U}_n + \{P(n\pi)^2 + P_0^2 (n\pi)^4\} U_n] \sin(n\pi x) = \\ \epsilon \left[-C_d \sum_{n=1}^{\infty} \dot{U}_n \sin(n\pi x) - \right. \\ \alpha_1 \sum_{n=1}^{\infty} U_n (n\pi)^2 \sin(n\pi x) \int_0^1 \sum_{L=1}^{\infty} U_L^2 L^2 \pi^2 \cos^2(L\pi x) dx + \alpha_2 [V_{DC} + \\ V_{AC} \cos(\Omega t)]^2 \left(1 + 2 \sum_{n=1}^{\infty} U_n \sin(n\pi x) + \right. \\ \left. 3 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{j=0}^{\infty} U_n U_m \frac{4}{(2j+1)\pi} \sin(n\pi x) \sin(m\pi x) \sin((2j+1)\pi x) \right) \left. \right] + \\ O(\epsilon^2), \end{aligned} \quad (12)$$

Now multiplying equation (12) with $\sin(k\pi x)$ where $k \in N$ for fixed k and integrate

with respect to x from 0 to 1, we get the integral term is defined as follow:

$$\int_0^1 \sin(n\pi x) \sin(k\pi x) dx = \begin{cases} 0, & k \neq n, \\ \frac{1}{2}, & k = n \end{cases} \quad (13)$$

$$\begin{aligned} \int_0^1 \sin(n\pi x) \sin(m\pi x) \sin((2j+1)\pi x) \sin(k\pi x) dx &= 2 \sum_{k=2j+1+m-n}^{\infty} - \\ 2 \sum_{k=n-m-(2j+1)}^{\infty} &+ \sum_{k=2j+1-n-m}^{\infty} + \sum_{k=n+m+2j+1}^{\infty} - \\ \sum_{k=n+m-(2j+1)}^{\infty} &, \end{aligned} \quad (14)$$

The coupled system of in-finitely many ordinary differential equations is obtained:

$$\begin{aligned} \ddot{U}_k + \omega_k^2 U_k &= \epsilon \left[-C_d \dot{U}_k - \frac{\alpha_1}{2} U_k (k\pi^2)^2 (\sum_{L=1}^{\infty} L^2 U_L^2) + 2\alpha_2 [V_{DC} + \right. \\ V_{AC} \cos(\Omega t)]^2 &\left(\frac{2}{(2k-1)\pi} + U_k + \frac{3}{2\pi} [2 \sum_{k=2j+1+m-n}^{\infty} - \right. \\ 2 \sum_{k=n-m-(2j+1)}^{\infty} &+ \sum_{k=2j+1-n-m}^{\infty} + \sum_{k=n+m+2j+1}^{\infty} - \\ \sum_{k=n+m-(2j+1)}^{\infty} &\left. \left. \left(\frac{U_n U_m}{(2j+1)} \right) \right] \right] + O(\epsilon^2), \end{aligned} \quad (15)$$

The system of ODE's is given in Eq. (15) is not easy to handle analytically. Therefore, to find the solution of Eq. (15), two timescales perturbation method is utilized. The solution of Equation (15) is supposed in the form, $U_k(x, t) = W_k(x, t_0, t_1; \epsilon)$ where, $t_0 = t$ (Fast – scale), $t_1 = \epsilon t$ (Slow – scale), $0 < \epsilon \ll 1$ and equating the coefficient of ϵ and ϵ^0 to zero. We have $O(1)$ and $O(\epsilon)$ problem, respectively

$$\frac{\partial^2 W_{k0}}{\partial t_0^2} + \omega_k^2 W_{k0} = 0, \quad (16)$$

$$\begin{aligned} \frac{\partial^2 W_{k1}}{\partial t_0^2} + \omega_k^2 W_{k1} &= -2 \frac{\partial^2 W_{k0}}{\partial t_0 \partial t_1} - C_d \frac{\partial W_{k0}}{\partial t_0} - \frac{\alpha_1}{2} W_{k0} (k\pi^2)^2 (\sum_{L=1}^{\infty} L^2 W_{L0}^2) + \\ 2\alpha_2 [V_{DC} + V_{AC} \cos(\Omega t)]^2 &\left(\frac{2}{(2k-1)\pi} + W_{k0} + \frac{3}{2\pi} [2 \sum_{k=2j+1+m-n}^{\infty} - \right. \\ 2 \sum_{k=n-m-(2j+1)}^{\infty} &+ \sum_{k=2j+1-n-m}^{\infty} + \sum_{k=n+m+2j+1}^{\infty} - \\ \sum_{k=n+m-(2j+1)}^{\infty} &\left. \left. \left(\frac{W_{n0} W_{m0}}{(2j+1)} \right) \right] \right). \end{aligned} \quad (17)$$

The solution of Eq. (16) is:

$$W_{k0}(t_0, t_1) = A_{k0}(t_1) \cos(\omega_k t_0) + B_{k0}(t_1) \sin(\omega_k t_0), \quad (18)$$

Where $\omega_k^2 = P(k\pi)^2 + P_0(k\pi)^4$ is a natural frequency of the beam system.

Substituting equation (18) in the right side of equation (17), we obtain

$$\begin{aligned} \frac{\partial^2 W_{k1}}{\partial t_0^2} + \omega_k^2 W_{k1} = & -2 \left[-\frac{dA_{k0}}{dt_1} \sin(\omega_k t_0) + \frac{dB_{k0}}{dt_1} \cos(\omega_k t_0) \right] \omega_k - \\ & C_d [-A_{k0} \sin(\omega_k t_0) + B_{k0} \cos(\omega_k t_0)] \omega_k - \frac{\alpha_1}{4} (k\pi^2)^2 [A_{k0} \cos(\omega_k t_0) + \\ & B_{k0} \sin(\omega_k t_0)] \sum_{L=1}^{\infty} L^2 (A_{L0}^2 + B_{L0}^2) - \frac{\alpha_1}{8} (k\pi^2)^2 \sum_{L=1}^{\infty} L^2 (B_{L0}^2 - \\ & A_{L0}^2) (A_{k0} (\cos((2\omega_L + \omega_k)t_0) + \cos((2\omega_L - \omega_k)t_0)) + B_{k0} (\sin((2\omega_L + \\ & \omega_k)t_0) - \sin((2\omega_L - \omega_k)t_0)) - \frac{\alpha_1}{4} (k\pi^2)^2 \sum_{L=1}^{\infty} L^2 A_{L0} B_{L0} (A_{k0} (\sin((2\omega_L + \\ & \omega_k)t_0) + \sin((2\omega_L - \omega_k)t_0)) + B_{n0} (\cos((2\omega_L + \omega_k)t_0) - \cos((2\omega_L - \\ & \omega_k)t_0)) + 2\alpha_2 \left(V_{DC}^2 + \frac{V_{AC}^2}{2} \right) \left[\frac{2}{(2k-1)\pi} + A_{k0} \cos(\omega_k t_0) + B_{k0} \sin(\omega_k t_0) + \right. \\ & \left. \frac{3}{2\pi} \left[2 \sum_{k=2j+1+m-n}^{\infty} - 2 \sum_{k=n-m-(2j+1)}^{\infty} + \sum_{k=2j+1-n-m}^{\infty} + \right. \right. \\ & \left. \left. \sum_{k=n+m+2j+1}^{\infty} - \sum_{k=n+m-(2j+1)}^{\infty} \right] \left(\frac{1}{(2j+1)} \right) \left[\cos((\omega_n + \omega_m)t_0) \left\{ \frac{A_{n0}A_{m0}}{2} - \right. \right. \right. \\ & \left. \left. \frac{B_{n0}B_{m0}}{2} \right\} + \cos((\omega_n - \omega_m)t_0) \left\{ \frac{A_{n0}A_{m0}}{2} + \frac{B_{n0}B_{m0}}{2} \right\} + \sin((\omega_n + \omega_m)t_0) \left\{ \frac{A_{n0}B_{m0}}{2} + \right. \right. \\ & \left. \left. \frac{A_{m0}B_{n0}}{2} \right\} + \sin((\omega_n - \omega_m)t_0) \left\{ \frac{A_{m0}B_{n0}}{2} - \frac{A_{n0}B_{m0}}{2} \right\} \right] \right] + \\ & 4\alpha_2 V_{DC} V_{AC} \left(\frac{1}{(2k-1)\pi} \cos(\Omega t_0) + \frac{A_{k0}}{2} \{ \cos((\omega_k + \Omega)t_0) + \cos((\omega_k - \Omega)t_0) \} + \right. \\ & \left. \frac{B_{k0}}{2} \{ \sin((\omega_k + \Omega)t_0) + \sin((\omega_k - \Omega)t_0) \} + \frac{3}{4\pi} \left[2 \sum_{k=2j+1+m-n}^{\infty} - \right. \right. \\ & \left. \left. 2 \sum_{k=n-m-(2j+1)}^{\infty} + \sum_{k=2j+1-n-m}^{\infty} + \sum_{k=n+m+2j+1}^{\infty} - \right. \right. \\ & \left. \left. \sum_{k=n+m-(2j+1)}^{\infty} \right] \left(\frac{1}{(2j+1)} \right) \left[\frac{1}{2} \{ \cos((\omega_n + \omega_m + \Omega)t_0) + \cos((\omega_n + \omega_m - \right. \right. \\ & \left. \left. \Omega)t_0) \} \left\{ \frac{A_{n0}A_{m0}}{2} - \frac{B_{n0}B_{m0}}{2} \right\} + \frac{1}{2} \{ \cos((\omega_n - \omega_m - \Omega)t_0) + \cos((\omega_n - \omega_m + \right. \right. \\ & \left. \left. \Omega)t_0) \} \left\{ \frac{A_{n0}A_{m0}}{2} + \frac{B_{n0}B_{m0}}{2} \right\} + \frac{1}{2} \{ \sin((\omega_n + \omega_m + \Omega)t_0) + \sin((\omega_n + \omega_m - \right. \right. \\ & \left. \left. \Omega)t_0) \} \left\{ \frac{A_{n0}B_{m0}}{2} + \frac{A_{m0}B_{n0}}{2} \right\} + \frac{1}{2} \{ \sin(\omega_n - \omega_m + \Omega)t_0 + \sin(\omega_n - \omega_m - \right. \right. \\ & \left. \left. \Omega)t_0 \} \left\{ \frac{A_{m0}B_{n0}}{2} - \frac{A_{n0}B_{m0}}{2} \right\} \right] \right] + \alpha_2 V_{AC}^2 \left(\frac{1}{(2k-1)\pi} \cos(2\Omega t_0) + \frac{A_{k0}}{2} \{ \cos((\omega_k + \right. \\ & \left. 2\Omega)t_0) + \cos((\omega_k - 2\Omega)t_0) \} + \frac{B_{k0}}{2} \{ \sin((\omega_k + 2\Omega)t_0) + \sin((\omega_k - 2\Omega)t_0) \} + \right. \\ & \left. \frac{3}{4\pi} \left[2 \sum_{k=2j+1+m-n}^{\infty} - 2 \sum_{k=n-m-(2j+1)}^{\infty} + \sum_{k=2j+1-n-m}^{\infty} + \right. \right. \end{aligned} \quad (19)$$

$$\begin{aligned} & \left[\sum_{k=n+m+2j+1}^{\infty} - \sum_{k=n+m-(2j+1)}^{\infty} \right] \left(\frac{1}{(2j+1)} \right) \left[\frac{1}{2} \{ \cos((\omega_n + \omega_m + 2\Omega)t_0) + \right. \\ & \cos((\omega_n + \omega_m - 2\Omega)t_0) \} \left\{ \frac{A_{n0}A_{m0}}{2} - \frac{B_{n0}B_{m0}}{2} \right\} + \frac{1}{2} \{ \cos((\omega_n - \omega_m - 2\Omega)t_0) + \\ & \cos((\omega_n - \omega_m + 2\Omega)t_0) \} \left\{ \frac{A_{n0}A_{m0}}{2} + \frac{B_{n0}B_{m0}}{2} \right\} + \frac{1}{2} \{ \sin((\omega_n + \omega_m + 2\Omega)t_0) + \\ & \sin((\omega_n + \omega_m - 2\Omega)t_0) \} \left\{ \frac{A_{n0}B_{m0}}{2} + \frac{A_{m0}B_{n0}}{2} \right\} + \frac{1}{2} \{ \sin((\omega_n - \omega_m + 2\Omega)t_0) + \\ & \sin((\omega_n - \omega_m - 2\Omega)t_0) \} \left\{ \frac{A_{m0}B_{n0}}{2} - \frac{A_{n0}B_{m0}}{2} \right\} \left. \right] \Bigg), \end{aligned}$$

Case 1: Non-Resonant Case ($\Omega \neq \omega_k$)

In this case the frequency of harmonic voltage Ω is not equal/ close to the natural frequency of the micro-beam that is $\Omega \neq \omega_k$ in the non-resonant case.

$$\begin{aligned} \frac{dA_{k0}}{dt_1} &= -\frac{C_d}{2} A_{k0} + \frac{\alpha_1}{16\omega_k} (k\pi^2)^2 B_{k0} [k^2 (A_{k0}^2 + B_{k0}^2) + 2 \sum_{L=1}^{\infty} L^2 (A_{L0}^2 + \\ & B_{L0}^2)] - \frac{\alpha_2}{\omega_k} \left(V_{DC}^2 + \frac{V_{AC}^2}{2} \right) B_{k0}, \\ \frac{dB_{k0}}{dt_1} &= -\frac{C_d}{2} B_{k0} + \frac{\alpha_1}{16\omega_k} (k\pi^2)^2 A_{k0} [k^2 (A_{k0}^2 + B_{k0}^2) + 2 \sum_{L=1}^{\infty} L^2 (A_{L0}^2 + \\ & B_{L0}^2)] - \frac{\alpha_2}{\omega_k} \left(V_{DC}^2 + \frac{V_{AC}^2}{2} \right) A_{k0}, \end{aligned} \quad (20)$$

The solution of the above system is

$$\begin{aligned} A_{k0} &= C e^{-\frac{C_d}{2} t_1} \cos \left(\frac{\alpha_1}{16\omega_k} (k\pi^2)^2 \left[k^2 \frac{C e^{-C_d t_1}}{C_d} + 2 \sum_{L=1}^{\infty} L^2 \frac{C e^{-C_d t_1}}{C_d} \right] + \right. \\ & \left. \frac{\alpha_2}{\omega_k} \left(V_{DC}^2 + \frac{V_{AC}^2}{2} \right) t_1 \right), \\ B_{k0} &= C e^{-\frac{C_d}{2} t_1} \sin \left(\frac{\alpha_1}{16\omega_k} (k\pi^2)^2 \left[k^2 \frac{C e^{-C_d t_1}}{C_d} + 2 \sum_{L=1}^{\infty} L^2 \frac{C e^{-C_d t_1}}{C_d} \right] + \right. \\ & \left. \frac{\alpha_2}{\omega_k} \left(V_{DC}^2 + \frac{V_{AC}^2}{2} \right) t_1 \right), \end{aligned} \quad (21)$$

By putting equation (21) into equation (18), we have the solution

$$\begin{aligned} U_k(t) &= C e^{-\frac{C_d}{2} t_1} \sin \left(\frac{\alpha_1}{16\omega_k} (k\pi^2)^2 \left[k^2 \frac{C e^{-C_d t_1}}{C_d} + 2 \sum_{L=1}^{\infty} L^2 \frac{C e^{-C_d t_1}}{C_d} \right] + \right. \\ & \left. \frac{\alpha_2}{\omega_k} \left(V_{DC}^2 + \frac{V_{AC}^2}{2} \right) t_1 \right) \sin(\omega_k t_0) + C e^{-\frac{C_d}{2} t_1} \cos \left(\frac{\alpha_1}{16\omega_k} (k\pi^2)^2 \left[k^2 \frac{C e^{-C_d t_1}}{C_d} + \right. \right. \\ & \left. \left. 2 \sum_{L=1}^{\infty} L^2 \frac{C e^{-C_d t_1}}{C_d} \right] + \frac{\alpha_2}{\omega_k} \left(V_{DC}^2 + \frac{V_{AC}^2}{2} \right) t_1 \right) \cos(\omega_k t_0). \end{aligned} \quad (22)$$

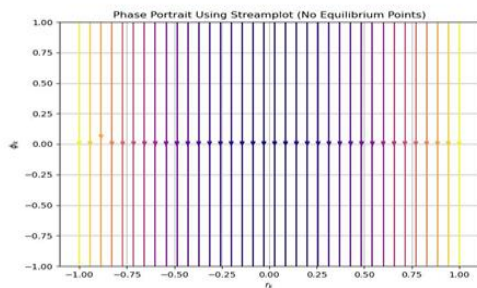
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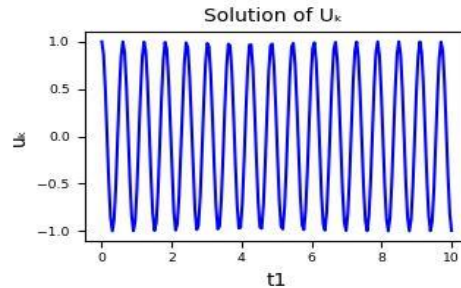
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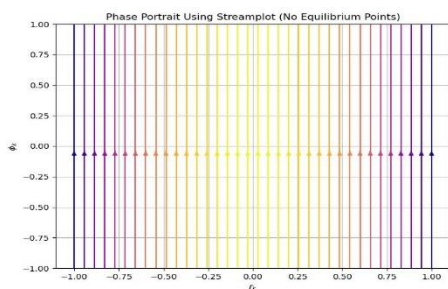


(a)

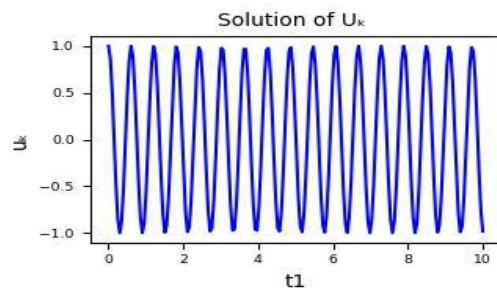


(b)

Fig 2: $\alpha_1 = 0.00001, \alpha_2 = 0.00001, c_d = 0, V_{DC} = 0.0001, V_{AC} = 0.0001$.

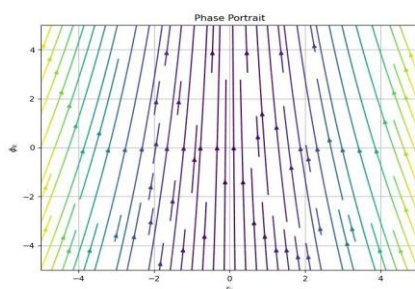


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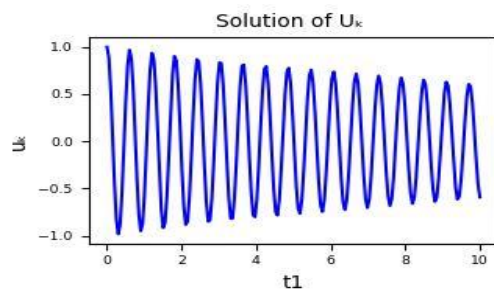


(b)

Fig 3: $\alpha_1 = 0.00001, \alpha_2 = 0.00001, c_d = 0, V_{DC} = 1000, V_{AC} = 0.0001$.



(a)



(b)

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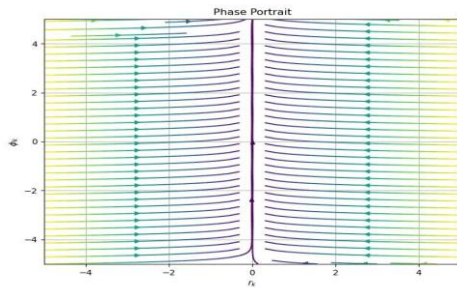
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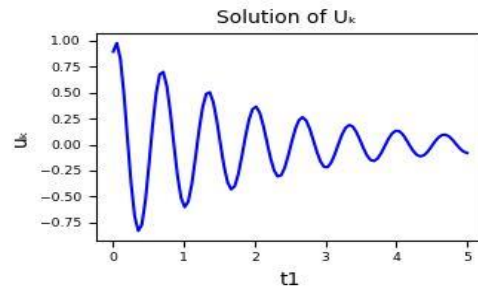
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Fig 4: $\alpha_1 = 0.00001, \alpha_2 = 0.00001, c_d = 0.1, V_{DC} = 1000, V_{AC} = 0.0001$.

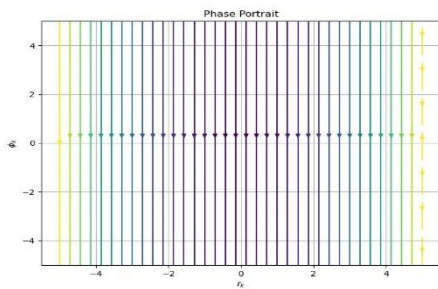


(a)

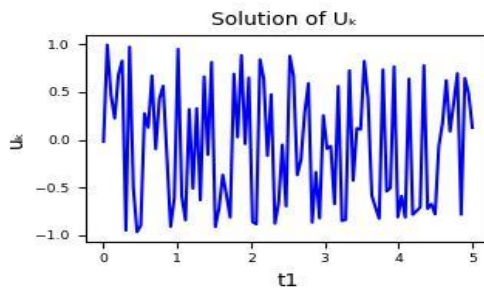


(b)

Fig 5: $\alpha_1 = 0.00001, \alpha_2 = 0.00001, c_d = 1, V_{DC} = 1000, V_{AC} = 0.0001$.

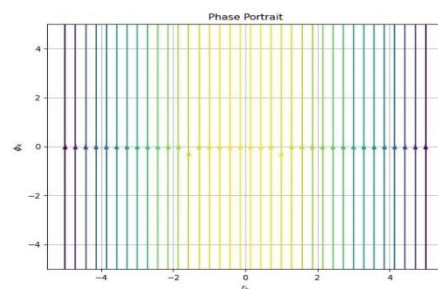


(a)

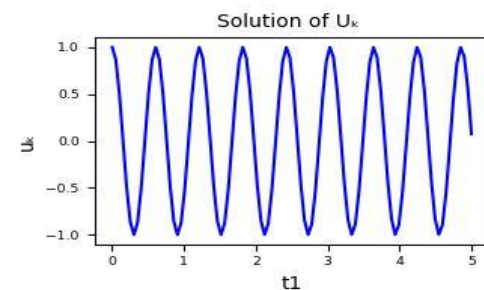


(b)

Fig 6: $\alpha_1 = 100, \alpha_2 = 0.00001, c_d = 0, V_{DC} = 0.0001, V_{AC} = 0.0001$.



(a)



(b)

Online ISSN

3007-3197

Print ISSN

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Fig 7: $\alpha_1 = 0.00001, \alpha_2 = 0.00001, c_d = 0, V_{DC} = 0.0001, V_{AC} = 100$.

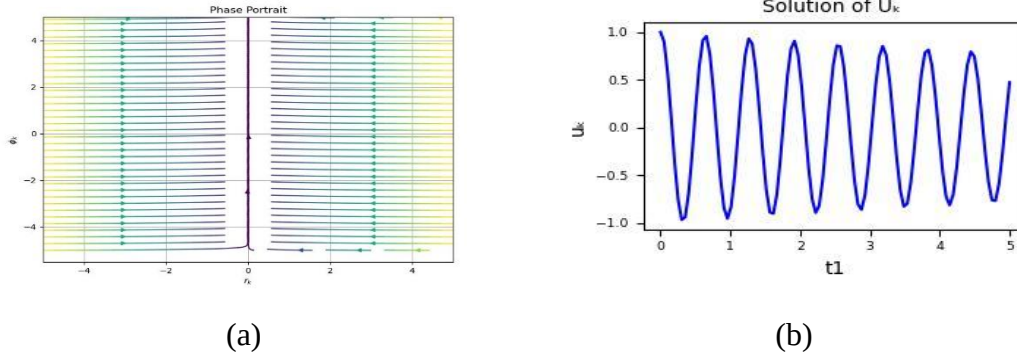


Fig 8: $\alpha_1 = 0.00001, \alpha_2 = 0.00001, c_d = 0.1, V_{DC} = 0.0001, V_{AC} = 100$.

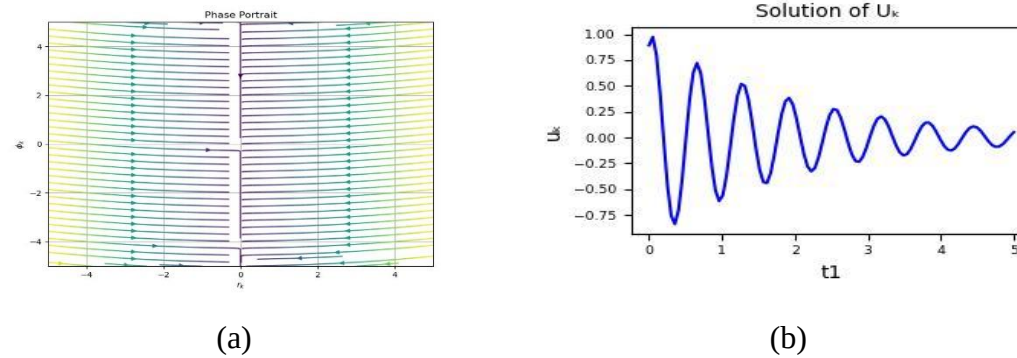


Fig 9: $\alpha_1 = 0.00001, \alpha_2 = 0.00001, c_d = 1, V_{DC} = 0.0001, V_{AC} = 100$.

In the non-resonant case, the behavior of the microbeam system under the effect of various parameters is exhibited in time vs amplitude graphs (Fig (a) 2-9) and phase portraits (Fig (b) 2-9). It is found out that the amplitude response remains bounded under the effect of AC/DC voltage and damping parameter.

Case 2: Resonant Case ($\Omega \neq \omega_k$)

The resonance cases are considered to examine the stability of microbeam: $\Omega = 0, \Omega = \varepsilon\sigma$. In this case, the frequency of harmonic voltage is equal or close to the zero of the natural frequency of micro beam that is the $\Omega \approx 0$. In order to represent the nearness of Ω with natural frequency of the system, we introduce the following detuning

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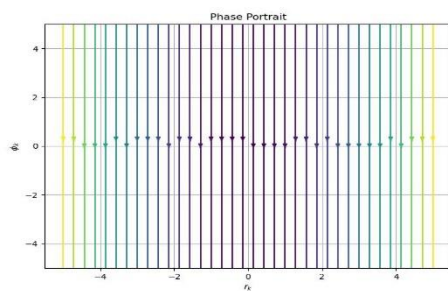
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parameter σ that is $\Omega = \varepsilon\sigma$, where $\sigma = O(1)$ and $0 < \varepsilon \ll 1$ in the resonance case.

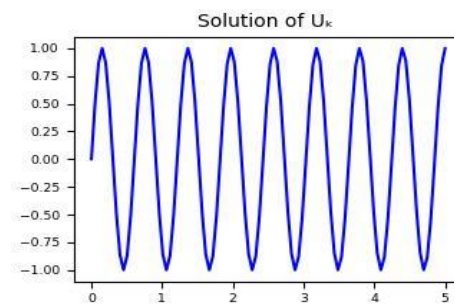
$$\begin{aligned} \frac{dA_{k0}}{dt_1} &= -\frac{c_d}{2}A_{k0} + \frac{\alpha_1}{16\omega_k}(k\pi^2)^2 B_{k0}[k^2(A_{k0}^2 + B_{k0}^2) + 2\sum_{L=1}^{\infty} L^2(A_{L0}^2 + \\ &B_{L0}^2)] - \frac{\alpha_2}{\omega_k}\left(V_{DC}^2 + \frac{V_{AC}^2}{2}\right) - 2\frac{\alpha_2}{\omega_k}V_{DC}V_{AC}B_{k0}\cos(\varepsilon\sigma t_0) - \\ &\frac{\alpha_2}{2\omega_k}V_{AC}^2 B_{k0}\cos(2\varepsilon\sigma t_0), \\ \frac{dB_{k0}}{dt_1} &= -\frac{c_d}{2}B_{k0} - \frac{\alpha_1}{16\omega_k}(k\pi^2)^2 A_{k0}[k^2(A_{k0}^2 + B_{k0}^2) + 2\sum_{L=1}^{\infty} L^2(A_{L0}^2 + \\ &B_{L0}^2)] + \frac{\alpha_2}{\omega_k}\left(V_{DC}^2 + \frac{V_{AC}^2}{2}\right) + 2\frac{\alpha_2}{\omega_k}V_{DC}V_{AC}A_{k0}\cos(\varepsilon\sigma t_0) + \\ &\frac{\alpha_2}{2\omega_k}V_{AC}^2 A_{k0}\cos(2\varepsilon\sigma t_0), \end{aligned} \quad (23)$$

In order to find solution, the above coupled system of ODEs, we will introduce the polar coordinate as $A_{k0} = r_k(t_1)\cos(\varphi_k(t_1))$ and $B_{k0} = r_k(t_1)\sin(\varphi_k(t_1))$ so from (23) we have:

$$\begin{aligned} \frac{dr_k}{dt_1} &= -\frac{c_d}{2}r_k, \\ \frac{d\varphi_k}{dt_1} &= -\frac{\alpha_1}{16\omega_k}(k\pi^2)^2[k^2r_k^2 + 2\sum_{L=1}^{\infty} L^2r_k^2] + \frac{\alpha_2}{\omega_k}\left(V_{DC}^2 + \frac{V_{AC}^2}{2}\right) + \\ &\frac{2\alpha_2}{\omega_k}V_{DC}V_{AC}\cos(\sigma t_1) + \frac{\alpha_2}{2\omega_k}V_{AC}^2\cos(2\sigma t_1). \end{aligned} \quad (24)$$



(a)



(b)

Online ISSN

3007-3197

Print ISSN

3007-3189

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Fig 10: $\alpha_1 = 0.00001, \alpha_2 = 0.00001, c_d = 1, V_{DC} = 0.0001, V_{AC} = 0.0001$.

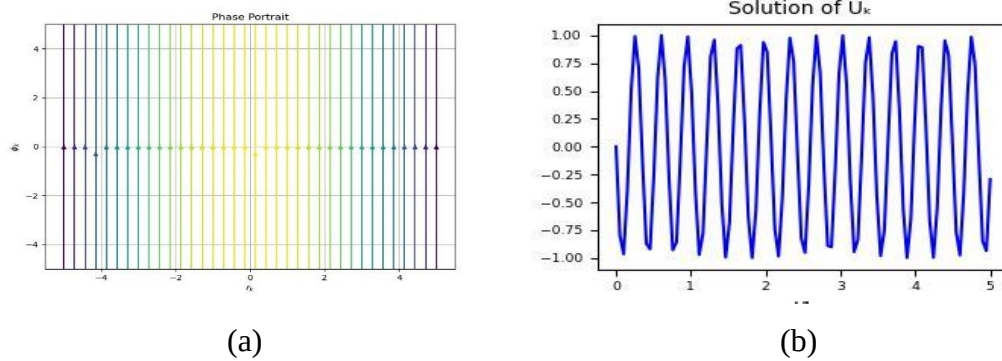


Fig 11: $\alpha_1 = 0.00001, \alpha_2 = 0.00001, c_d = 0, V_{DC} = 1000, V_{AC} = 0.0001$.

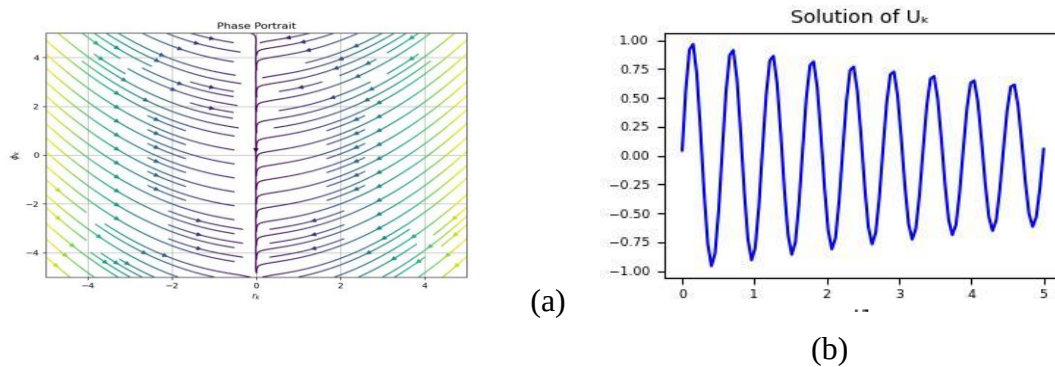
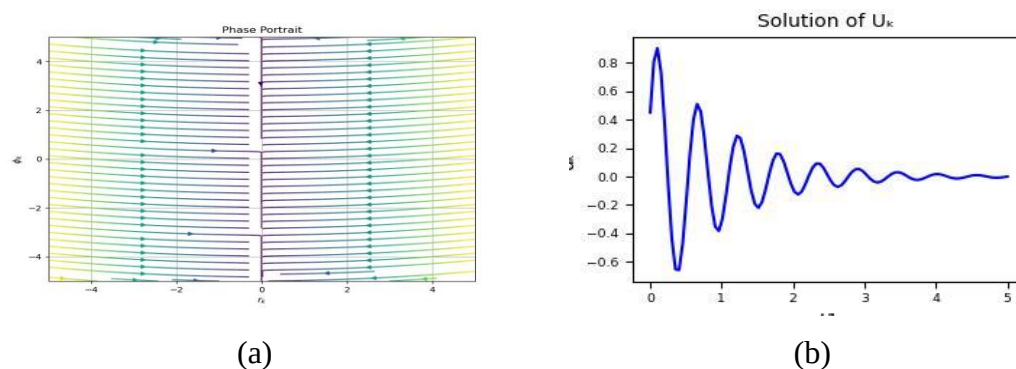


Fig 12: $\alpha_1 = 0.00001, \alpha_2 = 0.00001, c_d = 0.1, V_{DC} = 1000, V_{AC} = 0.0001$.



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3007-3197

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3007-3189

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Fig 13: $\alpha_1 = 0.00001, \alpha_2 = 0.00001, c_d = 1, V_{DC} = 1000, V_{AC} = 0.0001$.

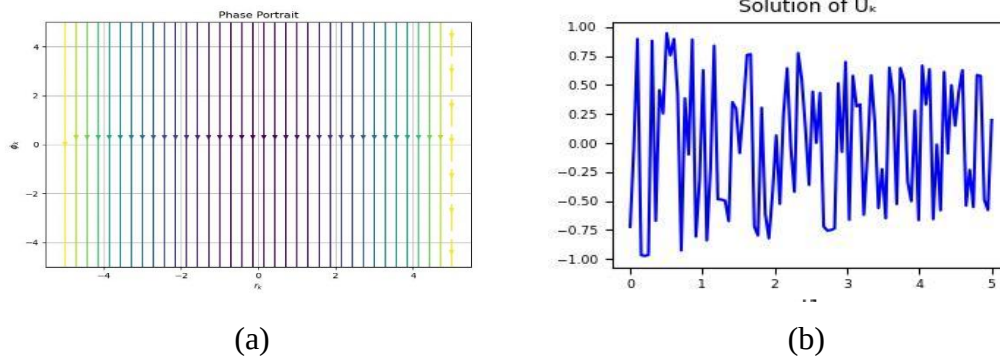


Fig 14: $\alpha_1 = 100, \alpha_2 = 0.00001, c_d = 0.1, V_{DC} = V_{AC} = 0.0001$.

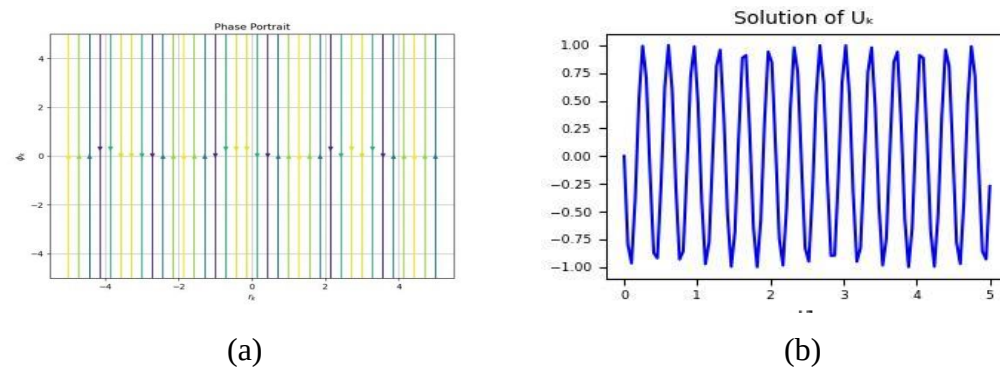


Fig 15: $\alpha_1 = 0.00001, \alpha_2 = 0.00001, c_d = 0, V_{DC} = 0.0001, V_{AC} = 100$.

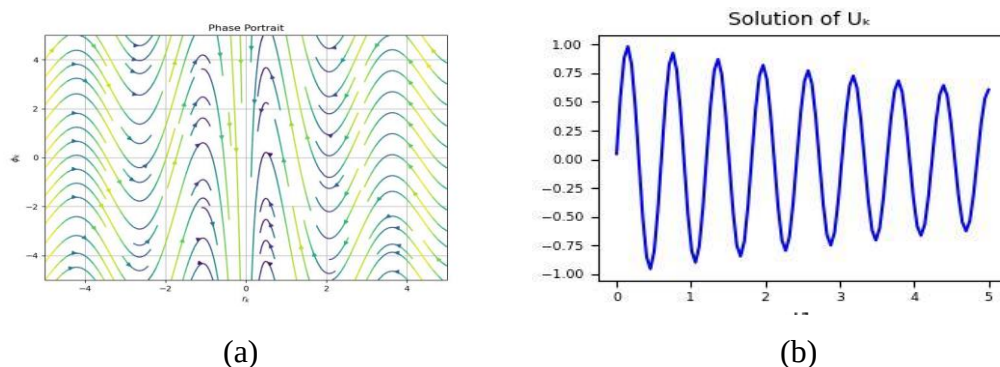


Fig 16: $\alpha_1 = 0.00001, \alpha_2 = 0.00001, c_d = 0.1, V_{DC} = 0.0001, V_{AC} = 100$.

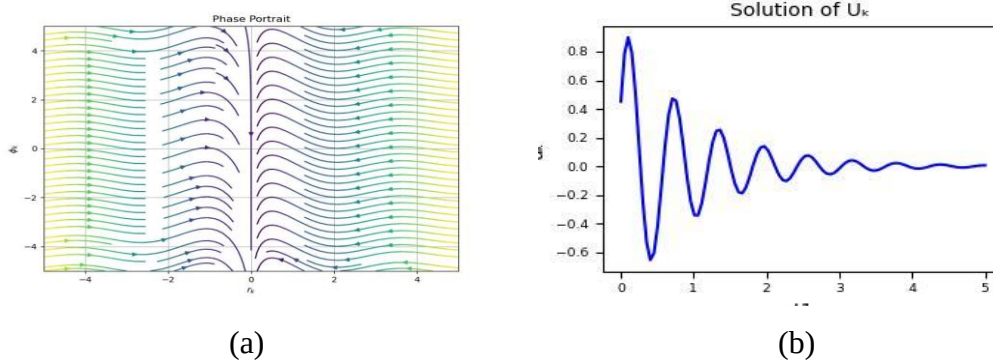


Fig 17: $\alpha_1 = 0.00001, \alpha_2 = 0.00001, c_d = 1, V_{DC} = 0.0001, V_{AC} = 100$.

In this resonant case, the behavior of the microbeam system under the effect of various parameters is exhibited in time vs amplitude graphs (Figure (a) 10-17) and phase portraits (Figure (b) 10-17). It is found out that the amplitude response remains bounded under the effect of α_1 , that is, if the values of α_1 decrease, the system is bounded and unbounded if the parameters α_1 increases. In addition, if the values of AC/DC voltage decrease, the system yields stable behavior and unstable if the values of AC/DC voltage increase, and if we increase damping parameter the system is stable and if we decrease the system is unstable.

Case 3: Resonant Case ($\Omega \neq \omega_k$)

In this case there is an interaction between mode 2 and 3. We will therefore consider only the equation for $k = 2$ and $k = 3$ we assume all other modes having zero initial energy. This truncation is allowed [25]

$$\begin{aligned}
 \frac{A_{20}}{dt_1} &= -\frac{c_d}{2} A_{20} + \frac{\alpha_1}{2\omega_2} \pi^4 B_{20} [2(A_{20}^2 + B_{20}^2) + \sum_{L=1}^{\infty} L^2 (A_{L0}^2 + B_{L0}^2)] - \\
 &\frac{\alpha_2}{w_2} V_{DC}^2 B_{20} - \frac{24\alpha_2}{21\pi\omega_2} V_{DC}^2 (A_{20}B_{30} - B_{20}A_{30}), \\
 \frac{B_{20}}{dt_1} &= -\frac{c_d}{2} B_{20} - \frac{\alpha_1}{2\omega_2} \pi^4 A_{20} [2(A_{20}^2 + B_{20}^2) + \sum_{L=1}^{\infty} L^2 (A_{L0}^2 + B_{L0}^2)] + \\
 &\frac{\alpha_2}{w_2} V_{DC}^2 A_{20} + \frac{24\alpha_2}{21\pi\omega_2} V_{DC}^2 (A_{20}A_{30} + B_{20}B_{30}), \\
 \frac{A_{30}}{dt_1} &= \frac{c_d}{2} A_{30} + \frac{9\alpha_1}{16\omega_3} \pi^4 B_{30} [9(A_{30}^2 + B_{30}^2) + 2 \sum_{L=1}^{\infty} L^2 (A_{L0}^2 + B_{L0}^2)] -
 \end{aligned} \tag{25}$$

Online ISSN

3007-3197

Print ISSN

3007-3189

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$$\frac{\alpha_2}{w_3} V_{DC}^2 B_{30} - \frac{24\alpha_2}{21\pi\omega_3} V_{DC}^2 (2A_{20}B_{20}),$$

$$\frac{B_{30}}{dt_1} = -\frac{c_d}{2} B_{30} - \frac{9\alpha_1}{16\omega_3} \pi^4 A_{30} [9(A_{30}^2 + B_{30}^2) + 2 \sum_{L=1}^{\infty} L^2 (A_{L0}^2 + B_{L0}^2)] +$$

$$\frac{\alpha_2}{w_3} V_{DC}^2 A_{30} + \frac{24\alpha_2}{21\pi\omega_3} V_{DC}^2 (-A_{20}^2 + B_{20}^2),$$

To solve the coupled system of ODEs we introduce the polar coordinate $A_{k0} = r_k(t_1) \cos(\varphi_2(t_1))$ and $B_{k0} = r_k(t_1) \sin(\varphi_2(t_1))$. After substituting, we get the following system

$$\frac{dr_2}{dt_1} = -\frac{c_d}{2} r_2 - \frac{24\alpha_2}{21\pi\omega_2} V_{DC}^2 r_2 r_3 \sin(\varphi_3 - 2\varphi_2),$$

$$\frac{d\varphi_2(t_1)}{dt_1} = -\frac{\alpha_1}{2\omega_2} \pi^4 [2r_2^2 + \sum_{L=1}^{\infty} L^2 r_L^2] + \frac{\alpha_2}{\omega_2} V_{DC}^2 + \frac{24\alpha_2}{21\pi\omega_2} V_{DC}^2 r_3 \cos(\varphi_3 - 2\varphi_2),$$

$$\frac{dr_3}{dt_1} = -\frac{c_d}{2} r_3 + \frac{24\alpha_2}{21\pi\omega_3} V_{DC}^2 r_2^2 \sin(\varphi_3 - 2\varphi_2), \quad (26)$$

$$\frac{d\varphi_3(t_1)}{dt_1} = -\frac{9\alpha_1}{16\omega_3} \pi^4 [9r_2^2 + 2 \sum_{L=1}^{\infty} L^2 r_L^2] + \frac{\alpha_2}{\omega_3} V_{DC}^2 + \frac{24\alpha_2}{21\pi\omega_3} V_{DC}^2 \frac{r_2^2}{r_3} \cos(\varphi_3 - 2\varphi_2),$$

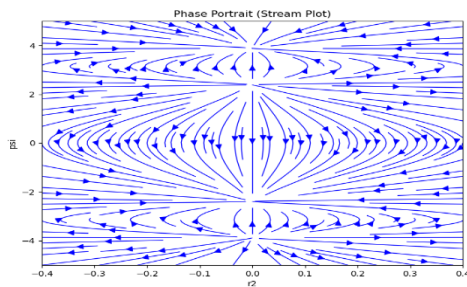


Fig 18: $\alpha_1 = 0.00001, \alpha_2 = 0.00001, c_d = 1, V_{DC} = 10000$.

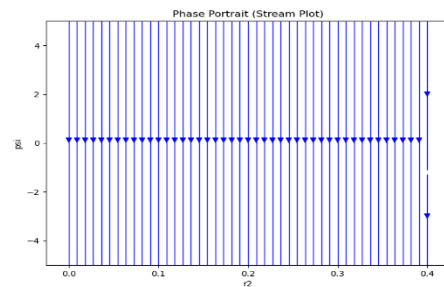


Fig 19: $\alpha_1 = 0.00001, \alpha_2 = 0.00001, c_d = 0, V_{DC} = 10000$.

In this resonant case, the behavior of the microbeam system under the effect of damping parameters is exhibited in the form of phase portraits (Figure 18-19). It is found that the values of AC/DC voltage and damping parameter are crucial for the stability of system.

5. Conclusion

Online ISSN

3007-3197

Print ISSN

3007-3189

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This study conducted a comprehensive stability analysis of a nonlinear damped microbeam governed by nonlinear (PDEs) of motion. The method of two time-scales perturbation, combined with Fourier series expansion, was employed to construct the approximate-analytic solution of the system. The Fourier expansion method discretized the spatial domain, enabling the application of perturbation techniques to the temporal dynamics, while the two timescales method used to determine the nonlinear interactions governing the microbeam's response. Through this approach, all possible resonance conditions-including ω_1 , ω_2 were systematically identified. The phase-portraits under these resonances were sketched, revealing critical insights into the system's stability.

The analysis demonstrated that the microbeam exhibits unstable behavior, characterized by unbounded amplitude growth, when damping is negligible. This instability is particularly pronounced at resonance frequencies, where energy transfer mechanisms dominate. However, the introduction of sufficient damping was shown to mitigate these effects, stabilizing the system by dissipating energy and suppressing resonant amplification. These findings underscore the pivotal role of damping in microbeam design, especially in microelectromechanical systems (MEMS), where minimizing instability risks is crucial for reliable operation. In addition, the impact of AC/DC on the system's response was also observed. It turned out that the system yields stable behavior when AC parameter α_1 increase and unstable behavior when α_1 decreases. Similar is the situation with DC parameter α_2 . This research highlights the necessity of incorporating adequate damping mechanisms and impact of AC/DC parameters in applications involving microbeams subjected to dynamic loads.

Data Availability

Data sharing is not applicable to this article, as all the data used to obtain the results is included within the article.

Conflicts of Interest

The authors declare no conflicts of interest.

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