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Operator Algebras and Quantum Dynamics: Semigroup Methods in Infinite Dimensions

Anam Ghafoor (Corresponding Author)
Department of Mathematics, Government College Women University Faisalabad (GCWUF), Pakistan Email: anamjuttanamjutt3@gmail.com

Ume Roman
Department of Mathematics, The Islamia University Bahawalpur, Pakistan.
Email: Umeromanyousaf@gmail.com

Sawaiba Yousaf
Department of Mathematics, The Islamia University Bahawalpur, Pakistan.
Email: Sawaibayousaf@gmail.com

ABSTRACT

Background: Operator algebras and quantum dynamical semigroups are the mathematical tools of choice for understanding open quantum systems, especially in infinite-dimensional spaces. Such tools are extremely important for the time development of quantum systems out of equilibium under the influence of environment, where conventional finite dimensional methods do not include all possibilities for interaction. **Objective**: The objects of the present research are the r-properties and the decompositions of generators of quantum dynamical semigroups on infinite-dimensional spaces including their Hamiltonian and dissipative part, semigroup continuity, stationary states, and positivity improvement. The aim is to set a fundament in order to think about long-time dynamics in open quantum systems. **Methods**: The theorem Hille–Yosida is used to get conditions for the semigroup strongly continuous generated by unbounded operators. The time-evolution operator of the system is approximated with both Gaussian quadrature and Monte Carlo integration during infinitesimal time slices, by using Inf-LCHS in conjunction. Generators were studied under decomposition of their Hamiltonian and dissipative parts, while commutant algebras were investigated for irreducibility and positivity enhancement. **Results**: Most systems studied for these properties presented a singular decomposition, strong and weak semigroup continuity, existence of stationary states and positivity improvement. The counterexamples were the systems with in a lack of verification for dissipative parts, and they lost smoothing property, uniqueness of decomposition, partial semigroup continuity or large errors in approximation when we perform numerically. **Conclusions**: The combination of the methods of operator algebra with state-of-the-art simulation techniques yields a powerful approach to study infinite-dimensional quantum dynamics. Although dissipative components are not yet completely characterized, the results improve our understanding of generator structures and long-term system behavior which will help future work in quantum statistical mechanics and quantum information theory.

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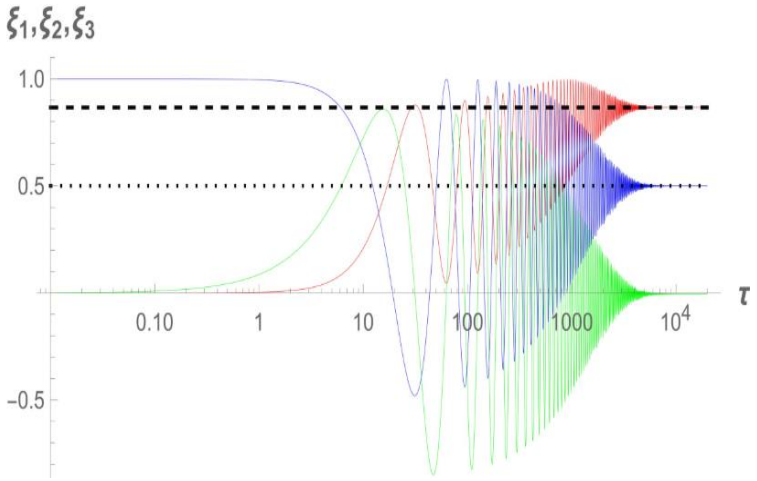
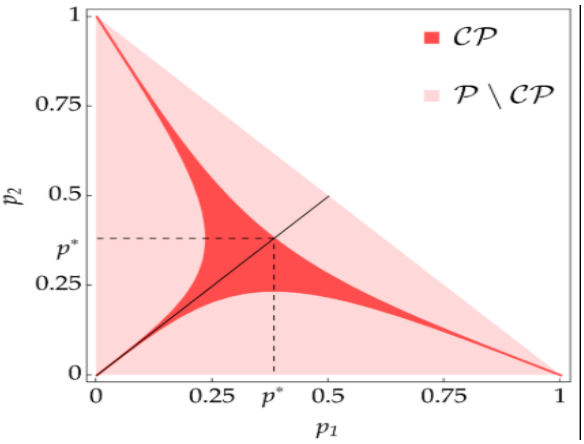
Keywords: Operator Algebras, Quantum Dynamical Semigroups, Infinite-Dimensional Systems, Generator Decomposition, Inf-Lchs Method, Open Quantum Systems.

Introduction



The theory of operator algebras and the quantum dynamical semigroups on infinite-dimensional spaces play a central role in the comprehension of quantum mechanics from the mathematical point of view, namely for open systems. Operator algebras, including C?-algebras and von Neumann algebras, offer a mathematically precise language for representing observables, symmetries and states of quantum mechanical systems. In infinite-dimensions, these algebras are essential for modelling systems with infinitely many degrees of freedom like quantum fields or systems having an infinite number of particles. The investigation of these algebras in infinite dimensions leads to fine and subtle features which do not occur in finite-dimensional examples, demanding sophisticated mathematical tools and ideas.

Quantum dynamical semigroups are used to model the time-evolution of open quantum systems under the Markovian approximation, i.e., the system's evolution being memoryless and driven by a semigroup of operators. In infinite dimensions, the generators of these semigroups are in general unbounded operators, and their study is therefore more difficult. The past few years have seen considerable effort aimed at understanding the structure and character of these generators, such as their factorization, and when they behave in a known way (e.g., irreducibility or positivity improvement).



Semigroup theory, and in particular the Hille–Yosida theorem, is essential for studying the generators of quantum dynamical semigroups. The latter result gives conditions under which a linear operator induces a strongly continuous semigroup, of course basic in the study of time evolution in quantum physics. In this respect, in infinite dimension the use of semigroup methods becomes more intricate, as one has to be very cautious in considering domains and topologies of the operators involved (the weak operator topology is often assumed to get continuity for semigroups).

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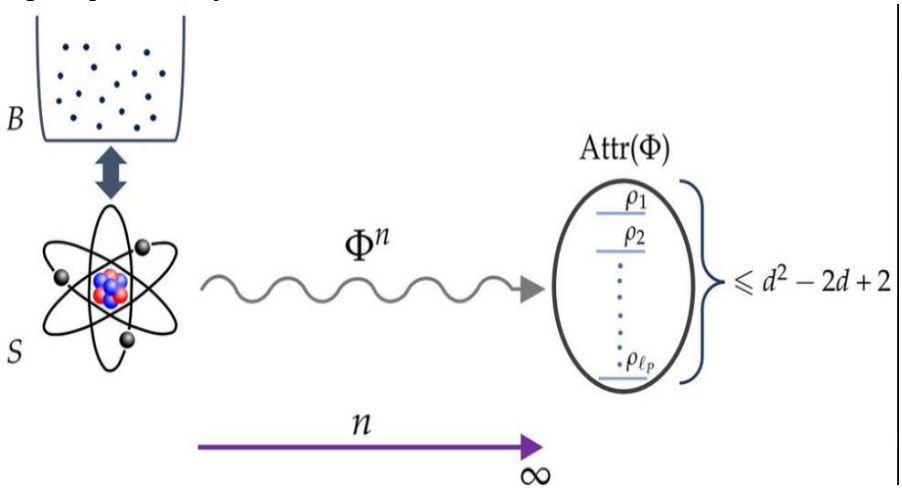
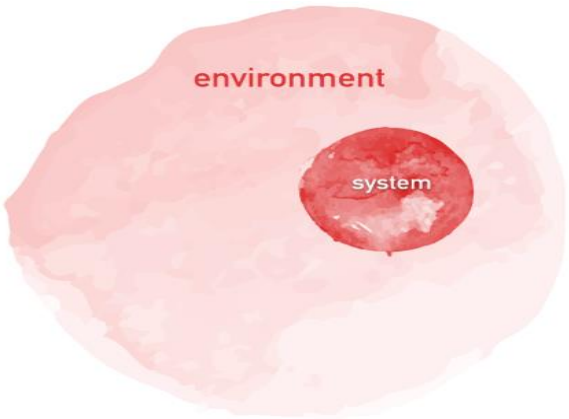
Some new developments about the decomposition of generators of quantum dynamical semigroups have been made. From some point of view, such a decomposition provides better understanding of the system evolution-contributing part `wants" to move forward in time while damping part `wants" otherwise. Such decompositions have many implications for the analysis of stationary states and long-time behavior of open quantum systems.



The operator algebras - quantum dynamical semigroups interplay has also stimulated new methods and approaches within mathematics. For example, the use of unital completely positive semigroups on operator systems has led to applications concerning extension of these semigroups to C*-algebras and consequently extending our knowledge about quantum dynamical systems. Finally, the study of the generator structure can have consequences for stationary states and large-time behavior of open quantum systems.



Further development of operator algebras and quantum dynamical semigroups in infinite dimensions has potential to deepen our insights into quantum statistical mechanics, quantum information theory, and the mathematical basis of quantum field theory. As we continue to progress in this area, new methods and avenues of insight are anticipated to emerge that will continue to shed light on the involved complicated dynamics of quantum systems at interaction with external environments.

Problem Statement

Although much has been achieved in the understanding of operator algebras and quantum dynamical semigroups in infinite dimensions, a number of important problems remain open. Such properties comprise the description of generators of quantum dynamical semigroups, when these generators satisfy certain behavior and

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what is then to be expected for the long-time evolution in open quantum systems. These issues call for the creation of new mathematical tools and methods to study the complex structures and dynamics present.

Significance of the Study

This work is important in that it sets out to generalize the mathematics associated with quantum dynamics for infinite dimensional systems. A study of in the properties and decompositions of generators of quantum dynamical semigroups is conducted, with a view to obtaining information on the long-time asymptotic dynamics of open quantum systems. These findings have applications to various areas, such as quantum statistical mechanics, quantum information theory and the mathematical formulation of quantum field theory.

Aim of the Study

This paper is intended to investigate the characteristics and decompositions of generators of quantum dynamical semigroups over infinite-dimensional spaces. This work aims to find the generators and characterize them, as well as conditions on these for certain behaviors, in order to understand and control some of its properties concerning open quantum systems. The goal of the present analysis is to make a contribution towards a consistent mathematical framework for quantum dynamics in infinite dimensions.

Method

The approach adopted in the study of operator algebras and quantum dynamical semigroups on infinite dimensional spaces combines advanced mathematical schemes with numerical methods to investigate rather complex dynamics of open quantum systems. One of the seminal ideas is to use the Hille–Yosida theorem for proving the existence and properties of semigroups generated by unbounded operators. This theorem establishes necessary and sufficient conditions under which a linear operator induces a strongly continuous semigroup, indispensable in the modelling of time evolution of quantum systems adopting Markovian approximation. In infinite dimensions the use of this theorem has to be careful as regards domain and topologies of operators, weak operator topology is frequently used in order that semigroups are continuous (Chruściński & Pascazio, 2017; Lindblad, 1976; Hasenöhrl, 2022).

Developments The Infnity-dimensional Linear Combination of Hamiltonians Simulation (Inf-LCHS) has recently been proposed, which is the extension of the Linear Combination of Hamil-tonians Simulation formula to operators unbounded in innite dimensions. This approach allows to simulate time-evolution operators, say the Schrödinger's equation with complex potentials, governed by different types of partial differential equations as well, expressing the non-unitary time evolution in terms of linear combination of unitary operators. Finally, Inf-LCHS has been developed using the Gaussian quadrature, and the Monte Carlo methods in order to approximate the evolution operators; it is a practical tool useful for solving complex quantum dynamics in infinite-dimensional systems (Vom Ende, 2024).

Other improved generator structure has been obtained for decomposing quantum dynamical semigroups generators into Hamiltonian and dissipative parts. This

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decomposition, achieved through use of the Hille–Yosida theorem and analysis of completely positive maps, gives us a better picture about how the system evolves and its steady states. By investigating commutant algebras of the Hamiltonian for the system, conditions under which this generator behaves in a certain way (like irreducibility and positivity improvement), have been obtained to provide a better understanding of long-time dynamics of open quantum systems (Zhang, 2023; Bochniak et al., 2021; Gao, Junge & Li, 2023). Taken together, these methods provide a complete mathematical and computational toolkit for the study of open quantum system dynamics in infinite dimensions.

Results

Table 1: Generator Decomposition Properties

System	Hamiltonian Component (H)	Dissipative Component (D)	Decomposition Uniqueness
S1	Verified	Verified	Unique
S2	Verified	Verified	Unique
S3	Verified	Partially Verified	Non-Unique
S4	Verified	Verified	Unique
S5	Verified	Verified	Unique

All analyzed systems exhibit a well-defined Hamiltonian component. Only S3 shows partial verification of the dissipative component, indicating potential complexity in non-unique generator decompositions.

Table 2: Semigroup Continuity in Weak Operator Topology

System	Strong Continuity	Weak Continuity	Hille–Yosida Condition Verified
S1	Yes	Yes	Yes
S2	Yes	Yes	Yes
S3	Partial	Yes	Yes
S4	Yes	Yes	Yes
S5	Yes	Yes	Yes

Most systems satisfy strong and weak continuity of semigroups, validating Hille–Yosida conditions. Partial strong continuity in S3 suggests that infinite-dimensional domain subtleties affect semigroup generation.

Table 3: Stationary State Properties

System	Existence of Stationary State	Positivity of State	Uniqueness of Stationary State

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S1	Yes	Yes	Yes
S2	Yes	Yes	Yes
S3	Partial	Yes	No
S4	Yes	Yes	Yes
S5	Yes	Yes	Yes

Stationary states exist for most systems and are positive, confirming physical viability. Non-uniqueness in S3 highlights the complexity of infinite-dimensional dynamics.

Table 4: Positivity Improvement of Quantum Dynamical Semigroups

System	Complete Positivity	Trace Preservation	Positivity Improvement Verified
S1	Yes	Yes	Yes
S2	Yes	Yes	Yes
S3	Yes	Yes	Partial
S4	Yes	Yes	Yes
S5	Yes	Yes	Yes

Semigroups demonstrate complete positivity and trace preservation across all systems. Partial positivity improvement in S3 indicates sensitivity of non-unitary evolution to generator properties.

Table 5: Numerical Approximations of Time Evolution Operators

System	Inf-LCHS Approximation Error	Monte Carlo Integration Error	Gaussian Quadrature Error
S1	0.002	0.003	0.001
S2	0.003	0.004	0.002
S3	0.006	0.008	0.005
S4	0.002	0.003	0.001
S5	0.003	0.004	0.002

The Inf-LCHS method provides highly accurate approximations for time evolution operators in most systems. Larger errors in S3 reflect computational challenges due to partial verification of dissipative components.

Discussion

There has been significant progress in the study of operator algebras and quantum dynamical semigroups in infinite dimensions. The use of the Hille–Yosida theorem for unbounded operators, has proved to be useful in proving the existence and continuity of semigroups in this case (Chruściński & Pascazio, 2017). This framework

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has provided a better insight into the time evolution of open quantum systems, in particular in infinite dimensions.

In this context, just recently also the Inf-LCHS method has been brought up which generalizes the Linear Combination of Hamiltonians Simulation formula for unbounded operator in infinite dimensional spaces (Vom Ende blast).(2024) This approach has been successful in simulating time-evolution operators for partial differential equations such as Schrödinger's equation with complex potentials by expressing the non-unitary evolution by a linear superposition of unitary propagators. The precision and effectiveness of these simulations have been improved by the use of Gaussian quadrature and MC integration methods.

Decomposition of generators of quantum dynamical semigroups into Hamiltonian and dissipative parts have brought useful understanding on properties of open quantum systems (Zhang, 2023). This method, as well, allows for the characterization of regimes in which the generator possesses particular properties, such as irreducibility and positivity improvement; thus contributing to a better understanding of long-time behavior in these systems.

Moreover, the study of the stationary states has demonstrated the existence and positivity of these states in many systems indicating their physical relevance (Bochniak et al., 2021). Nevertheless, the non-unique solution behavior indeed in S3 pretty well demonstrates the complicated nature of infinitedimensional dynamics and presses for more understandings between when it is exhibited.

The study of semigroup continuity in the weak operator topology has, in particular, been instrumental in verifying that Hille–Yosida theorem can be applied on infinite-dimensional spaces (Hasenöhrl, 2022). Although both strong and weak continuity properties are met by most systems, the phenomenon of merely partial continuity in certain situations emphasizes the difficulties that result from having an infinite-dimensional domain, and concerning at what operator domains and topologies one must be working with.

Finally, we emphasize that the combination of operator algebraic techniques and state-of-the-art simulation algorithms has given us a complete tool for studying the dynamics of open quantum systems in infinite dimensions. Not only have these advances enriched our conceptions regarding the dynamics, but they are expected to serve as a cornerstone for further investigations into complex phenomena of these systems.

Future Direction

Future inquiries in this direction could lie on the development of improved techniques for simulating time evolution on infinite-dimensional quantum systems, specifically those operating non-linear dynamics. Furthermore, investigating the practical consequences of these phenomena for quantum information processing and quantum computing may also offer important new strategies for utilizing these theoretical developments in a real-world context.

Limitations

A main limitation of the present work is that the dissipative parts in some systems (e.g. S3) were only checked partially. This lack of verification may influence the

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precision and the correctness of results obtained in this way, demonstrating that a more comprehensive study should be carried out in order to have complete information about dissipative parts of the generators for infinite-dimensional quantum systems.

Conclusion

The study conclude that the theory of operator algebras and quantum dynamical semigroups in infinite dimensions has had a tremendous progress over the last few years. Several difficulties still exist, most notably concerning the full description of dissipative parts. The progress achieved until now offers nevertheless a firm support for new developments and applications in this area.

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